Topological Properties of Dynamical Systems

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abstract *This report is composed of two parts: the first covering the mathematics and the second covering applications of ideas developed in the former. In the first part, we begin by a review of the topological classification of fixed points in linear dynamical systems (Arnol'd 1973). We then develop the basic apparatus of differential topology (Milnor 1972; Guillemin and Pollack 2010), and consider elementary results in homotopy theory (Munkres 2000). We finally establish the notions of degree and index and the corresponding classical results. The level of rigor varies throughout the first part with the main focus being on brevity and clarity rather than on completeness (left-out parts of proofs are documented nevertheless). In the second part we review a few related results , in the application of topological methods (Glass 1975; Glass 1977; Strogatz 1985; Winfree and Strogatz 1983, Winfree 2001), specifically degree and index theorems, to biological/chemical dynamics. The theme of this study is arguably the "impossibility of continuous functions" which is revisited multiple times in the first part and used extensively in the second part. Specifically, as we introduce various pieces of machinery, we prove variations of a non-retraction theorem.*

Contents

Part I: theory

Classification of linear fixed points

In this section we summarize the topological classification off fixed points in linear systems, following the analysis in (Arnol'd 1973). We will see that the language of this classification will also be useful for arbitrary systems and are closely related to the Poincaré-Hopf index theorem. Consider an arbitrary dynamical system, in the generic form $\dot{x} = F(x)$, for some $F : \mathbb{R}^n \to \mathbb{R}^n$ nice enough to have well defined solutions over the *phase space* $M \subseteq \mathbb{R}^n$. A *phase curve* is any trajectory $\{f^t, t \in \mathbb{R}\}$ in the phase space given some initial condition x and the *phase flow*, denoted by $\{f^t\}$, is the one-parameter group of all such t-advance mappings $f^t : M \to M$. Two phase flows $\{f^t\}, \{g^t\} : M \to M$ are said to be *equivalent*, with qualifications to follow, if there exists a one-to-one mapping $h : M \to M$ that carries each flow to the other, that is:

$$h \circ f^t = g^t \circ h$$

for any time $t \in \mathbb{R}$. If h is linear, we consider the two flows (and thus the corresponding systems) *linearly equivalent*. If h is a diffeomorphism (to be defined below), we consider the flows *differentiably equivalent*. And finally, if h is merely a homeomorphism, we consider the flows *topologically equivalent*. Hereon, we will focus on the latter kind of equivalence which is the weakest form of equivalence in the following sense:

Theorem: Linear equivalence implies differentiable equivalence and differentiable equivalence implies topological equivalence.

proof: Any linear bijective map is a diffeomorphism and any diffeomorphism is a homeomorphism. ■

In what follows we review the topological classification of fixed points of linear systems. Consider generic linear systems

$$\dot{x} = Ax \text{ , and } \dot{y} = By$$

where A and B are linear maps on $\mathbb{R}^n$. It turns out to be the only "interesting" questions lie in topological equivalence since:

Theorem: Two linear systems $\dot{x} = Ax$ and $\dot{y} = By$ over $\mathbb{R}^n$ are *linearly equivalent* if and only if the spectrum of A and B, i.e their set of eigenvalues, coincide.[1] Furthermore, they are *differentiably equivalent* if and only if the spectrum of A and B coincide.

proof: By definition, the systems are linearly equivalent iff there exists a one-to-one linear transformation $h : \mathbb{R}^n \to \mathbb{R}^n$ carrying the trajectories of one system to the other. Let this transformation be $y = Hx$ where H is the matrix representation of h in the standard basis. We have:

$$By = \dot{y} = H\dot{x} = HAx = HAH^{-1}y$$

and thus $B = HAH^{-1}$. Recalling the fact from linear algebra that the spectrum of HAH^{-1} coincides with that of A if and only if H is an isomorphism completes the proof for the first claim. We accept the second claim of the proposition as obvious. ■

No purely imaginary eigenvalues

The fundamental result in the topological classification of linear fixed points follows (Arnol'd 1973).

Remark: Eigenvalues of higher multiplicity introduce complications in proofs but do not change the correctness of the following statements.

Theorem (*Linear fixed point classification theorem*): Two linear systems $\dot{x} = Ax$ and $\dot{y} = Ay$, all of whose eigenvalues have nonzero real parts, are topologically equivalent if and only if they have the same number of eigenvalues with positive and negative real parts. Symbolically, A and B are topologically equivalent if and only if:

$$\pi(A) = \pi(B) \text{ , and } \mu(A) = \mu(B)$$

where π and μ count the number of eigenvalues with positive and negative real parts, respectively.

proof: Following (Arnol'd 1973)) we decompose the claim to the following three statements, the first of which is a known fact from linear algebra and the last two are fairly easy to prove.

Lemma: If a linear transformation $A : \mathbb{R}^n \to \mathbb{R}^n$ has no purely imaginary eigenvalues then the space $\mathbb{R}^n$ can be decomposed into an invariant direct sum

$$\mathbb{R}^n = \mathbb{R}^\pi + \mathbb{R}^\mu$$

where μ and π are the number of eigenvalues of A with negative and positive real parts, respectively.

Remark: When $\mu = 0$ the fixed point is an *unstable node*. When $\pi = 0$ the fixed point is a *stable node*. Otherwise the fixed point is a *saddle fixed point*. in which case we refer to $\mathbb{R}^\mu$ and $\mathbb{R}^\pi$ as the *incoming* and *outgoing* strands, respectively.

Lemma: If $A : \mathbb{R}^n \to \mathbb{R}^n$ is such that all its eigenvalues have positive real parts, then the linear system $\dot{x} = Ax$ is topologically equivalent to $\dot{x} = x$.

Remark: Similarly, if A is such that all its eigenvalues have negative real parts, then the linear system is topologically equivalent to $\dot{x} = -x$.

Lemma: If the four linear transformations $A_{1,2} : \mathbb{R}^n \to \mathbb{R}^n$ and $B_{1,2} : \mathbb{R}^m \to \mathbb{R}^m$ are such that $\dot{x} = A_1x$ is topologically equivalent to $\dot{x} = A_2x$ and that $\dot{y} = B_1y$ is topologically equivalent to $\dot{y} = B_2y$, then the product systems $(\dot{x}, \dot{y}) = (A_ix, B_iy)$ over $\mathbb{R}^{n+m}$ are topologically equivalent.

■

Purely imaginary eigenvalues

Topological classification of fixed points with purely imaginary eigenvalues runs into terrible difficulties with transcendental numbers. To demonstrate this, we consider a simple case: a decoupled product of two harmonic oscillators, with angular frequencies ω and ω'. The natural phase space for this system is the torus T^2. Suppose we wish to answer the following simple question: for what values of ω and ω' are

[1] and the simplicity of eigenvalues is immaterial.

the trajectories of the 4d system closed? Clearly if ω and ω' are rational multiples of each other (called rationally dependent, or commensurable) then the trajectories are closed. In fact, the following can be proved by using an elegant application of the pigeonhole principle (Dirichlet's principle) and a simple induction (Arnol'd 1973) (a separate induction will also prove an identical statement for n decoupled oscillators):

Theorem: Each individual trajectory of a system of two decoupled harmonic oscillators with angular frequencies ω and ω' is:

- closed on the torus if ω and ω' are rationally dependent, and
- everywhere dense on the torus otherwise.

And here is the difficulty: we do not have a solution for the commensurability theorem (and more generally the algebraic dependence problem). For example, it is not known whether π and e are commensurable or not.

Basics of differential topology

Terminology

Let $(X, \mathscr{T})$ be a topological space where $\mathscr{T}$ is, by definition, the collection of "open" sets in X. As usual, we drop $\mathscr{T}$ as long as the intended collection of open sets is clear from the context. Any subset $Y \subseteq X$ is a topological space of its own, referred to as the *subspace topology*, in which the collection of open sets is $\{U \cap Y; U \in \mathscr{T}\}$.[2] Therefore, we can unambiguously refer to any arbitrary (relatively) open subset of Y by $U \cap Y$ where $U \in \mathscr{T}$ is understood to be an open subset of the ambient topology. In what follows we will be mostly concerned with subsets of the euclidean space and therefore, the only topology we are concerned with is the standard topology on $\mathbb{R}^n$ or a subspace topology induced by the standard topology.

Notation

The half-space $H^n \subset \mathbb{R}^n$ is

$$H^n = \{(x_1, \ldots, x_n); x_n \geq 0\}$$

[2]Consider $\mathbb{R}$ with the standard topology and the induced subspace topology over $(0, 2]$. In this space $(1, 2]$ is an open subset and $(0, 1]$ is not. In general we have: open (closed) sets in the subspace topology of $Y \subseteq X$ are necessarily open (closed) sets in the ambient topology if and only if Y itself is open (closed) as a set in X.

The unit disk (closed unit ball) $D^n \subset \mathbb{R}^n$ is

$$D^n = \{(x_1, \ldots, x_n); \sum x_i^2 \leq 1\}$$

and the n-sphere $S^n \subset \mathbb{R}^{n+1}$ is

$$S^n = \{(x_1, \ldots, x_n, x_{n+1}); \sum x_i^2 = 1\} = \partial D^n$$

The product $S^1 \times S^1$ gives rise to the torus[3] T^2 and similarly the Cartesian product of n circles gives the n-torus $T^n \subset \mathbb{R}^{n+1}$. We will only be concerned with 2 dimensional tori in this report.

Given two topological spaces X and Y we say a function $f : X \to Y$ is a *homeomorphism* if it is a bicontinuous bijection; that is, a continuous bijection with a continuous inverse. If, additionally, f and f^{-1} have continuous partial derivatives of arbitrary order we call f a *diffeomorphism*. Naturally, diffeomorphic spaces are necessarily homeomorphic.

Example: Consider $\mathbb{R}$ with the standard topology and $[0, 1]$ with the subspace topology. The two are not homeomorphic since the latter is compact and the former is not. However, $\mathbb{R}$ and $(0, 1)$ are diffeomorphic, for example using a properly adjusted tangent function.

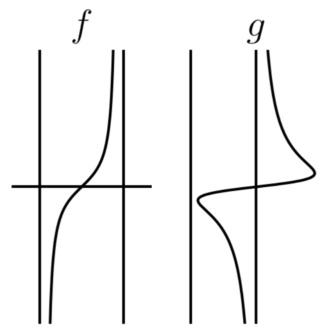


Figure 1: A diffeomorphism $f : (0, 1) \to \mathbb{R}$ and a non-homeomorphism $g : [0, 1] \to \mathbb{R}$

Manifolds

A subset M of a topological space X is a manifold of dimension m (or an m-manifold) if M is *locally homeomorphic* to the euclidean space $\mathbb{R}^m$; that is, for every $x \in M$ there exists a neighborhood $V \cap M$ of x in M and a homeomorphism g carrying an open set $U \subseteq \mathbb{R}^k$ to $V \cap M$. If, additionally, we demand that the local maps be diffeomorphisms (that is M is *locally diffeomorphic* to $\mathbb{R}^m$), we call M a *smooth m-manifold*.

[3]One can *define* the torus as this product or prove that the geometric object (defined as a parametrized surface) is homeomorphic to the product manifold of two circles.

In what follows, this is always the intended meaning of "manifold".

Example: The circle S^2 in the plane is a 2-manifold. To see this consider the two stereographic projections from the north pole p and the south pole q. The former is a diffeomorphism carrying $S^2 - \{p\}$ to $\mathbb{R}$ (which qualifies as an open subset of $\mathbb{R}$) and the latter does the same with $S^2 - \{q\}$. In fact, there is no way to map S^2 to $\mathbb{R}$ with less than two local diffeomorphisms since one is compact and the other is not. The same argument holds for S^n in $\mathbb{R}^n$.[4]

Example: The open unit ball in $\mathbb{R}^n$ is diffeomorphic to $\mathbb{R}^n$ and therefore a smooth n-manifold.

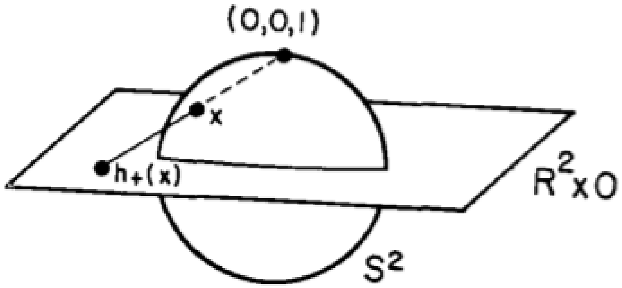


Figure 2: Stereographic projection (from (Milnor 1972))

Manifolds with boundary An important class of objects do not qualify as manifolds with the above definition, namely manifolds with boundary.

Theorem: The half-space H^n is not a manifold in $\mathbb{R}^n$.

proof: Consider any point on the boundary of H^n, that is any point $\mathbf{x} = (x_1, \ldots, x_{n-1}, 0)$. A neighborhood $U \cap H^n$ of $\mathbf{x}$ cannot be homeomorphic to an open subset of $\mathbb{R}^n$ since the neighborhood $U \cap H^n$, though open in H^n, is not open in $\mathbb{R}^n$ and thus cannot be homeomorphic to an open set in $\mathbb{R}^n$. ■

The half-space itself gives the basis for the next definition: $M \subset \mathbb{R}^k$ is a (smooth) m-*manifold with boundary* if it is locally diffeomorphic to the half-space H^m. We refer to the *boundary* of M, denoted by ∂M as the set of points in M corresponding to the boundary of H^m in the local maps.

Example: The unit disk D^n is locally diffeomorphic to H^n and therefore a smooth n-manifold with boundary. The local diffeomorphisms can be established by considering two inversion transformations centered one at each of the north and south poles and using the following two facts: **1** an inversion is diffeomorphic except at its center, and **2** the image of a sphere through the center is a hyperplane not containing the center.

Remark: The definition of the boundary is independent of the choice of local diffeomorphisms since, by the same argument as in the proposition above, no point on the boundary of H^n, which is the hyperplane with dimension $n-1$, can be mapped by a diffeomorphism to an interior point of M.

Remark: The boundary of a manifold in the above sense only coincides with the topological boundary when $m = k$. In fact, when $m < k$ all points in M are boundary points in the topological sense. In what follows we are only interested in the boundary in the manifold sense.

We need the following two results for further discussion. The proofs are straightforward and of the style of proofs in the next section.

Theorem: If M is a smooth manifold with boundary then ∂M is a smooth manifold without boundary and we have:

$$\dim \partial M = \dim M - 1$$

Furthermore, the interior $M - \partial M$ is a smooth manifold of the same dimension as M.

Theorem: If M and N are two smooth manifolds then $M \times N$ is also a smooth manifold. If M is a manifold with boundary and N is a manifold then $M \times N$ is a manifold with boundary.

Remark: The product of two manifolds with boundary is not necessarily a manifold (with or without boundary). For example, consider $[0,1]$ which is a 1-manifold in $\mathbb{R}$ with boundary $\{0,1\}$. The product $[0,1] \times [0,1]$ is not a manifold due to its corners.

Tangents and derivatives

Tangent spaces At every point x of an m-manifold M in $\mathbb{R}^k$ we can define the *tangent space* TM_x by means of the local diffeomorphism. Let g be a diffeomorphism carrying an open set $U \subseteq \mathbb{R}^m$ to a neighborhood $g(U)$ of x. Denote by $dg_x : \mathbb{R}^m \to \mathbb{R}^k$ the Jacobian of g which is a linear map represented, in standard coordinates, as a $k \times m$ matrix of partial derivatives of g. We now define TM_x to be the image of $\mathbb{R}^m$ under this linear map. That is:

$$TM_x = dg_x(\mathbb{R}^m) \subseteq \mathbb{R}^k$$

[4] Note that the unit sphere in a normed vector space V is compact if and only if $\dim V < \infty$. The "if" direction is established by the Heine-Borel theorem and the "only if" direction can be proved using Riesz's lemma.

In words, the tangent space to M at x is the image of $\mathbb{R}^m$ under the Jacobian of the local diffeomorphism evaluated at x.

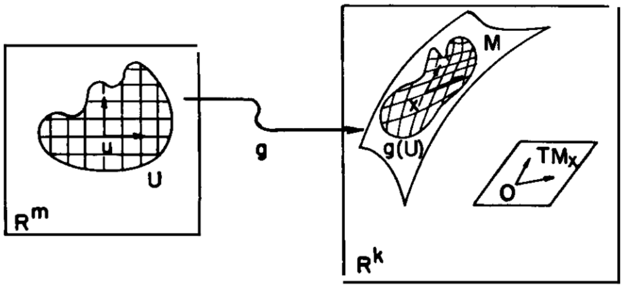


Figure 3: The tangent space of a manifold (from (Milnor 1972))

As usual, the intuition is that the shifted (affine) subspace $x + TM_x$ is the best approximation of M at x by a linear hyperplane of dimension $\dim M$. Once ambiguity that needs to be addressed is to demonstrate that this definition is independent of the choice of local parameterization g. We accept this as given, (Milnor 1972) and (Guillemin and Pollack 2010) both provide simple proofs.

Theorem: Let M be an m-manifold. Then TM_x has dimension m for any $x \in M$.

proof: At any point $x \in M$ there exists a local diffeomorphism g carrying an open set $U \subset \mathbb{R}^m$ to a neighborhood W of x. This gives the following commutative triangle:

$$\begin{tikzcd} & W \arrow[dr, "g^{-1}"] & \\ U \arrow[ur, "g"] \arrow[rr, "\mathrm{id}"] & & \mathbb{R}^m \end{tikzcd}$$

where id is the identity map. Differentiating gives:

$$\begin{tikzcd} & \mathbb{R}^k \arrow[dr, "d(g^{-1})_x"] & \\ \mathbb{R}^m \arrow[ur, "dg_u"] \arrow[rr, "\mathrm{id}"] & & \mathbb{R}^m \end{tikzcd}$$

from which it is clear that dg_x must have rank m. ∎

Derivative of maps on manifolds Given a map $f : M \to N$ between two manifolds M and N we can define the derivative df_x of f at a point $x \in M$, where $f(x) = y$, by means of the tangent spaces. The formal definition is more cumbersome than the geometric idea: df_x is the linear transformation carrying TM_x to TN_y. Suppose M is an m-manifold in $\mathbb{R}^k$ and N is an n-manifold in $\mathbb{R}^l$. There is a local diffeomorphism g carrying an open set $U \subset \mathbb{R}^m$ to a neighborhood $g(U)$ of x and a local diffeomorphism h carrying an open set $V \subset \mathbb{R}^n$ a neighborhood $h(V)$ of y. We can adjust U and V to get the the following commutative diagram:

$$\begin{tikzcd} g(U) \arrow[r, "f"] & h(V) \\ U \arrow[u, "g"] \arrow[r, "h^{-1} \circ f \circ g"'] & V \arrow[u, "h"'] \end{tikzcd}$$

which by demanding the chain rule gives the following commutative diagram (note that the derivatives of g and h are already well defined):

$$\begin{tikzcd} TM_x \arrow[r, "df_x"] & TN_y \\ \mathbb{R}^m \arrow[u, "dg_u"] \arrow[r, "d(h^{-1} \circ f \circ g)"'] & \mathbb{R}^l \arrow[u, "dh_v"'] \end{tikzcd}$$

Now we can simply define

$$df_x = dh_x \circ d(h^{-1} \circ f \circ g) \circ dg_x^{-1}$$

and be guaranteed to have a well defined derivative. By our construction, the following is obvious:

Theorem: If $f : M \to N$ is a diffeomorphism then $df_x : TM_x \to TN_y$ is an isomorphism everywhere. Consequently, M and N must have the same dimension.

The following theorem can be proved using the same procedure as above using local diffeomorphisms.

Theorem (*Inverse function theorem*): Let $f : M \to N$ be a smooth map on manifolds M and N both with the same dimension k and let $f(x) = y$. If df_x is an isomorphism (that is, a nonsingular linear map on $\mathbb{R}^k$) then there exists a neighborhood U of x in M that is mapped diffeomorphically to an open set $f(U)$ in N.

Classification of smooth 1-manifolds

We mention the following theorem in passing and with no proof as we will refer to it in the following sections ((Milnor 1972) and (Guillemin and Pollack 2010) both prove this in an appendix).

Theorem (*Classification of smooth 1-manifolds*): Any smooth, connected 1-dimensional manifold is either diffeomorphic to S^1 or to some interval in $\mathbb{R}$.

Corollary: The boundary of any smooth compact 1-manifold has an even number of points.

Regular values

In much of what follows regular values play in important role in establishing results that otherwise would require heavy machinery of algebraic topology. Here we define them and prove an important property. Consider a smooth map $f : M \to N$ where M and N have dimensions m and n respectively. For a point $x \in M$ with $f(x) = y$ we say that $x \in M$ is a *regular point* if df_x has rank n (equivalently, df_x is surjective).[5] We say that $y \in N$ is a *regular value* if $f^{-1}(y)$ contains only regular points. Correspondingly, we can define *singular points* in M and *singular values* in N.

We now establish a very useful result for the case where M is compact:

Theorem: Let $f : M \to N$ be a smooth map and suppose M is compact and $y \in N$ is a regular value. Then the set $f^{-1}(y) \subset M$ is finite.

proof: Since $\{y\}$ is a closed subset of N its preimage must be closed in M which by compactness of M implies $f^{-1}(y)$ must be compact. Furthermore, for any $x \in f^{-1}(y)$ there is a neighborhood of x in M over which f is one-to-one. Therefore $f^{-1}(y)$ is discrete; and being compact too, it must be finite. ■

For any regular value $y \in N$, we define $\# f^{-1}(y)$ to be the number of elements in $f^{-1}(y)$ (Milnor 1972).[6] From the above theorem it can be shown that: $\# f^{-1}(y)$ is *locally constant* over the set of regular values in N. We will return to this object later when we link the idea of homotopic equivalence classes to the degree of a mapping.

We end this introductory section by mentioning three important results without proof (see Guillemin and Pollack 2010 and Milnor 1972 for proofs):

Theorem (*Sard's theorem*): Let $f : U \to \mathbb{R}^n$ be a smooth map over an open set $U \subseteq \mathbb{R}^m$. Regular values of f are everywhere dense in $\mathbb{R}^n$ (in other words, the set of critical values of f has Lebesgue measure zero).

Theorem (*Preimage theorem*): Let $f : M \to N$ be a smooth map between smooth manifolds of dimension $m \geq n$. If $y \in N$ is a regular value of f then $f^{-1}(y)$ is a smooth $(m-n)$-manifold.

Theorem (*Preimage theorem for manifolds with boundary*): Let M be a manifold with boundary of dimension M and N be a manifold with dimension n where $m > n$. Suppose $f : M \to N$ is a smooth map. If $y \in N$ is a regular value of f *and* a regular value of f restricted to ∂M then $f^{-1}(y)$ is a smooth $(m-n)$-manifold with boundary and its boundary is precisely the intersection $f^{-1}(y) \cap M$.

Non-retraction theorem I

We are now ready to prove our first variation of the *non-retraction* theorem. This elegant proof was first given by M. Hirsch (Milnor 1972):

Theorem (*Non-retraction theorem I*): Let M be a compact manifold with boundary ∂M. There exists no smooth map $f : M \to \partial M$ which leaves ∂M pointwise fixed.

proof: Suppose there was such a map f. Let y be a regular value of f (the existence of many such points is guaranteed by Sard's theorem). Notice that f restricted to ∂M is the identity map and therefore by the preimage theorem for manifolds with boundary $f^{-1}(y)$ must be a smooth 1-manifold with boundary

$$\partial f^{-1}(y) = f^{-1}(y) \cap \partial M = \{y\}$$

But since the 1-manifold $f^{-1}(y)$ is also compact, by the classification theorem, it must have an even number of boundary points. Therefore, $\partial f^{-1}(y)$ must have an even number of elements which is a contradiction. ■

Corollary: Using the above theorem, Milnor gives an elegant proof of *Brouwer's fixed point theorem* (Milnor 1972) which says that any map $g : D^n \to D^n$ must have a fixed point. The idea of the proof is this. Supposing otherwise, for any x the two points $g(x)$ and x are distinct points in D^n and thus one can define the intersection of the line connecting them and S^{n-1}. This then introduces a smooth map from $f : D^n \to S^{n-1}$ leaving all points on S^{n-1} pointwise fixed which is a contradiction.

Basics of Homotopy Theory

Let X and Y be any topological spaces and f and f' be continuous functions from X into Y. We say that $H : X \times [0,1] \to Y$ is a *homotopy* (and that f and f' are *homotopic*) if it continuously deforms f into f' in the following

[5] In the case where $m = n$ the condition becomes: x is a regular point if df_x is an isomorphism on $\mathbb{R}^m$.

[6] The definition only makes sense when M is compact, which implies $f^{-1}(y)$ is finite.

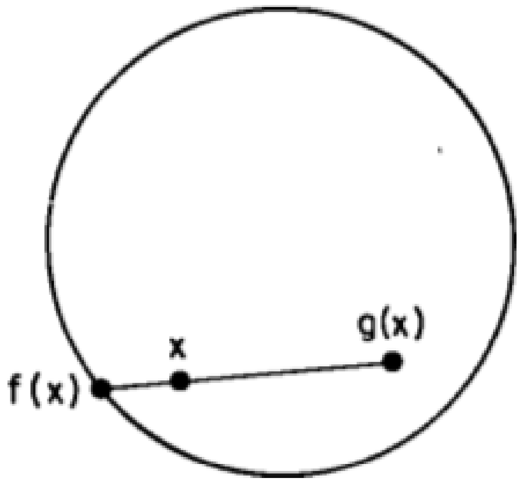


Figure 4: Milnor's proof of Brouwer's fixed point theorem (Milnor 1972)

sense: H is continuous and

$$H(x,0) = f(x)\ ,\ \text{and}\ \ H(x,1) = f'(x)$$

for all $x \in X$.

Similarly, if f and f' are smooth functions and we demand that H is a smooth function, then H is a *smooth homotopy* and we say that f and f' are *smoothly homotopic*.

Example: Any two continuous functions from $\mathbb{R}$ to $\mathbb{R}$ are homotopic.

Example: A *path* in a topological space X is a continuous functions $f : [0,1] \to X$ "connecting" to points in X, namely $f(0)$ and $f(1)$. Any two paths on $\mathbb{R}^n$ are homotopic.

Example: A *loop* in a topological space X is a path f satisfying $f(0) = f(1)$. More appropriately, a loop is a continuous function $f : S^1 \to X$. Any two loops in $\mathbb{R}^n$ are homotopic.

The above examples give the appearance that homotopies are not informative objects. We will see in the next section that this is not the case and that, in the above examples, homotopies are "trivial" merely because $\mathbb{R}^n$ is "trivial" in a certain sense: $\mathbb{R}^n$ is simply connected.

Before we proceed, we establish the most central property of homotopies, their composability: if H_1 is a homotopy from f_1 to f_2 and H_2 is a homotopy from $f2$ to f_3 then we can define $H = H_2 \circ H_1 : X \times [0,1] \to Y$ to be the *concatenation* of the two deformation processes, that is:

$$H(x) = \begin{cases} H_1(x, 2t) & \text{if } t < \frac{1}{2} \\ H_2(x, 1-2t) & \text{if } t \geq \frac{1}{2} \end{cases}$$

where continuity is guaranteed by the fact that

$$H(x, \frac{1}{2}) = H_1(x,1) = H_2(x,0) = f_2(x)$$

From this it follows that:

Theorem: The homotopy relation is an equivalence relation, henceforth denoted by $\simeq$.

This equivalence relation be the link connecting topology to group theory.

The same composability argument applies to arbitrary paths in a topological space: If $f, g : S^1 \to X$ are two loops in X then the composition $f * g$ is defined as the concatenation of the two loops in the same fashion as above. The resulting function is continuous only if $f(1) = g(0)$. If we denote by $\mathscr{P}$ the set of all paths over X then we can easily verify that the $*$ operation satisfies all properties of a group over $\mathscr{P}$, minus being applicable to all members of $\mathscr{P}$ (Munkres 2000)[7] with the identity element being the constant function and the inverse element corresponding to a path f being the path going backwards from the ending point of f to the starting point of f.

The Fundamental Group

We have already shown that the reason we did not get a proper group using the path concatenation operation $*$ is that it only applies to paths that are compatible (one ends where the other starts). The natural restriction that gives us a proper group is to only focus on loops: Let X be any topological space and $x_0 \in X$ be an fixed arbitrary point. Let $\mathscr{L}$ be the collection of all loops on X through x_0:

$$\mathscr{L} = \left\{ f : S^1 \to X; f \text{ cont's, and } \ x_0 \in f(S^1) \right\}$$

Then $(\mathscr{L}, *)$ is a group (Munkres 2000). Now consider the effect of the homotopy equivalence relation $\simeq$ on $\mathscr{L}$: we call the quotient space, that is loops through x_0 up to homotopy, the *fundamental group of X based at x_0* and denote it by $\pi_1(X, x_0)$. In other words, The homotopy relation $\simeq$ divides $\mathscr{L}$ into equivalence classes each containing a collection of homotopic loops through x_0.

Example: The fundamental group of the circle is $\mathbb{Z}$ the correspondence being the "winding number" of a loop about the circle. The group is generated by two elements: a single winding clockwise loop and a single winding counterclockwise loop.

[7] We can say, therefore, that $(\mathscr{P}, *)$ is a *groupoid* but this is not terribly interesting.

Two natural questions arise regarding the fundamental group:

1. Is the group $\pi_1(X, x_0)$ independent of x_0, at least up to isomorphism?
2. One guaranteed element of $\pi_1(X, x_0)$ is the set of loops homotopic to the constant loop $f(t) = x_0$. When is the this the only element of the fundamental group?

Question 1 can be answered by the following intuitively obvious theorem that we accept without proof (Munkres 2000):

Theorem: If X is a *path-connected* topological space the fundamental group is independent of the choice of base point up to isomorphism.

From here on, we will restrict our attention to path-connected spaces for which the above proposition holds. This allows us to drop the base point qualifier and unambiguously refer to the fundamental group of X by $\pi_1(X)$.[8]

It is easy to verify the following (Munkres 2000):

Theorem: For any two path-connected spaces X and Y we have:

$$\pi_1(X \times Y) \cong \pi_1(X) \times \pi_1(Y)$$

Example: By the proposition above the fundamental group of the torus $T^2 = S^1 \times S^1$ is, up to isomorphism, $\mathbb{Z}^2$. This should be obvious given our discussion of purely imaginary eigenvalues in linear systems. An induction provides us with the general relation $\pi_1(T^n) = \mathbb{Z}^n$.

Question 2 turns out to define its own topological property. We say that a topological space X is *simply connected* if it is path connected and if every loop $f : S^1 \to X$ through it can be *contracted* to a point in the sense that it is homotopic to the constant function. It is then a matter of definition, that:

Theorem: A simply connected topological space X has a trivial fundamental group.

Example: $\mathbb{R}^n$ is simply connected for any n. $\mathbb{R}^n - \{0\}$ is simply connected except for $n = 1, 2$(Munkres 2000). This is intuitively obvious: in $\mathbb{R}$ and $\mathbb{R}^2$ loops that contain the origin cannot be contracted to a point but in $\mathbb{R}^3$ (and higher dimensions) we have "extra dimensions" to deform a closed loop (which is a 1-manifold) around the origin and contract it to a point.

[8] This also frees us from having to work with *pointed spaces* which are symbolic representation of the following: if $f : X \to Y$ and $f(x_0) = y_0$ for some $x_0 \in X$ and $y_0 \in Y$ we refer to (X, x_0) as a pointed space and symbolically write: $f : (X, x_0) \to (Y, y_0)$.

Example: S^n is simply connected except for $n = 1$. This one can be proved as a consequence of the above which we will do in the section of deformation retractions.

Induced homomorphisms

The central result of this section is the following:

Theorem: Let X and Y be path-connected topological spaces.[9] Then any continuous function $h : X \to Y$ *induces a homomorphism* of fundamental groups: $h_* : \pi_1(X) \to \pi_1(Y)$.

proof: The homomorphism is given by $h_*(f) = h \circ f$ or more accurately $h_*([f]) = [h \circ f]$ where $[f]$ is the homotopy class of f. Checking that h_* respects the group operation $*$ of loop concatenation is straightforward. ■

From this two powerful corollaries follow:

Corollary: A *homeomorphism* between topological spaces *induces an isomorphism* of fundamental groups.

Corollary: Fundamental groups are a *topological invariant* in that they remain identical, up to isomorphism, under homeomorphisms.

Deformation Retractions

In this section we prove our second variation of the non-retraction theorem. Given a topological space X we call a continuous map $h : X \times [0, 1] \to X$ a *deformation retraction* onto a subspace A of X if $h(x, 0) = x$ and $h(x, 1) \in A$ for all $x \in X$ and $h(a, t) = a$ for all $a \in A$ and all $t \in [0, 1]$. In words, h is a deformation retraction if it is homotopy between the identity on X and a retraction on A leaving all members of A pointwise fixed throughout the deformation.

Theorem: If there exists a deformation retraction of X onto $A \subset X$ then the inclusion map $i : A \to X$ induces an isomorphism of fundamental groups. By this we mean, the induced homomorphism i_* of fundamental groups is bijective.

proof: The proof is rather difficult for arbitrary spaces (Munkres 2000). We provide the intuitive idea here: suppose X is $\mathbb{R}^{n+1} - \{0\}$ and A is S^n (for example S^1 as a subspace of $\mathbb{R}^2 - \{0\}$). The induced homomorphism i_* of the inclusion map is one that maps a loop f in S^n to a loop in $\mathbb{R}^{n+1} - \{0\}$ by not doing anything to it! The loop is already

[9] This restriction is not necessary (Munkres 2000) but simplifies the notation as discussed earlier.

in $\mathbb{R}^{n+1} - \{0\}$. To prove that this is an isomorphism of fundamental groups, we must show that all loops in $\mathbb{R}^{n+1} - \{0\}$ are deformation retractable to S^n (that is, they are homotopic to a loop lying entirely in S^n). This can be trivially done by collapsing all points in $\mathbb{R}^{n+1}$ continuously and radially onto S^n via $g(x) = x/|x|$ which is continuous everywhere in $\mathbb{R}^{n+1} - \{0\}$. ■

Example: We can now prove that S^n is simply connected for $n > 1$: consider the same function g as we used in the proof of the proposition. This is a deformation retraction from $\mathbb{R}^{n+1} - \{0\}$ to S^n which induces an isomorphism of fundamental groups. But the fundamental group of the $\mathbb{R}^{n+1} - \{0\}$ is trivial for $n + 1 > 2$ and thus S^n is simply connected for $n > 1$.

The propositions in the preceding two sections are extremely useful in that they allows us to prove the impossibility of continuous functions between topological spaces. We will make extensive use of them in the second part.

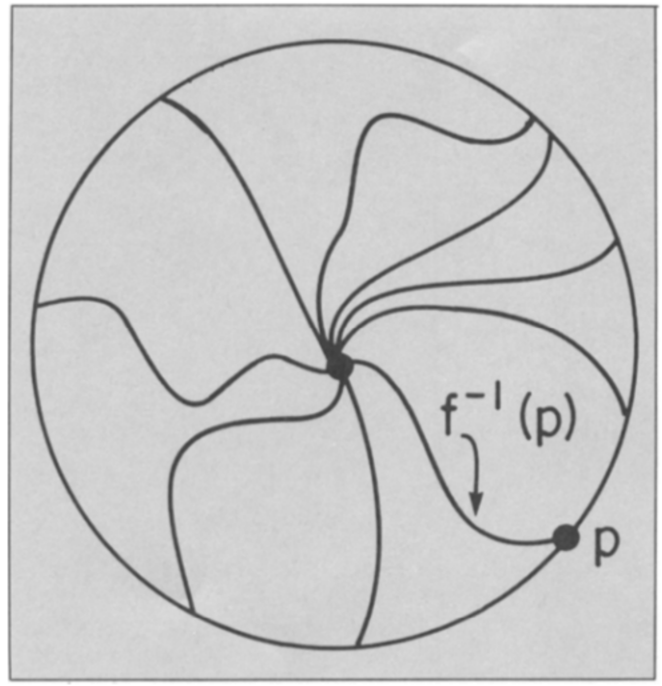


Figure 5: There is no deformation retraction from $D^2 \to S^1$ (Strogatz 1985)

Non-retraction theorem II

Theorem (*Non-retraction theorem II*): There exists no deformation retraction from D^2 to S^1.

proof: We know the fundamental group of S^1 is $\mathbb{Z}$ and that D^2 is simply connected and thus has a trivial fundamental group. Any deformation retraction of D^2 onto S^1 the two fundamental groups induces an isomorphism of the fundamental groups which is impossible since $\mathbb{Z} \ncong \{0\}$. ■

Remark: With all its glory, this method does not generalize to higher dimensions! We cannot use it to prove that D^{n+1} does not have a deformation retraction onto S^n for $n > 1$. This is because the fundamental group of S^n also becomes trivial for $n > 1$ and the contradiction disappears. Even in the exercises of (Munkres 2000) the n-dimensional case is delegated to degree theory which we will investigate in the next section.

Degree theorems

Degree mod 2

The results of this section rely crucially on the assumption that M is *compact and without boundary* and that M and N have the same dimension. We have already seen in previous sections that in such cases, for any regular value of f the set $f^{-1}(y)$ is finite and that the the number of points in it, denoted by $\#f^{-1}(y)$, is locally constant as y ranges over the regular values of f. We will see that under certain conditions degree mod 2 is a smooth-homotopy invariant and is independent of the choice of regular value y. The central lemma is this (Milnor 1972):

Theorem: (*Homotopy lemma*) Let $f, g : M \to N$ be smoothly homotopic maps between manifolds with equal dimension and suppose M is compact and without boundary. If $y \in N$ is a regular value of both M and N then

$$\#f^{-1}(y) \equiv \#g^{-1}(y) \mod 2$$

proof: Let the homotopy be $h : M \times [0, 1] \to N$. Since M is without boundary its product with $[0, 1]$ is a manifold with boundary. If y is a regular value of h as well, since it is also a regular value of f and g, by the preimage theorem for manifolds with boundary, the set $h^{-1}(y)$ is a smooth 1-manifold with boundary $\partial h^{-1}(y)$ equal to

$$h^{-1}(y) \cap [M \times 0 \cup M \times 1]$$

which is

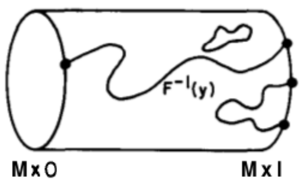
$$f^{-1}(y) \times 0 \cup g^{-1}(y) \times 1$$

It follows that the number of points in $\partial h^{-1}(y)$ is equal to

$$\#f^{-1}(y) + \#g^{-1}(y)$$

But since a compact 1-manifold has an even number of boundary points it follows that $\#f^{-1}(y) \equiv \#g^{-1}(y) \mod 2$.

If y is not a regular value of h then we will find a neighborhood $U \subseteq N$ of y consisting only of regular values of f over which $\#f^{-1}(y)$ is constant. Similarly, we can find a neighborhood $V \subseteq N$ consisting only of regular values of g over which $\#g^{-1}(y)$ is constant. Now by Sard's theorem, there must be a regular value of h within $U \cap V$ which would imply $\#f^{-1}(y) = \#g^{-1}(y)$. ■



The number of boundary points on the left is congruent to the number on the right modulo 2

Figure 6: Homotopy lemma (Milnor 1972)

If additionally we require that N is connected, then we can prove the following which we will accept without proof:

Theorem: Let $f : M \to N$ be a smooth map between manifolds of the same dimension. Further assume that M is compact with no boundary and that N is connected. Then for any two regular values $y, z \in N$ of f we have:

$$\#f^{-1}(y) \equiv \#f^{-1}(z) \mod 2$$

and this common value, called the *degree mod 2 of f* is identical to that of any map g which is smoothly homotopic to f.

Non-retraction theorem III

Theorem (*Non-retraction theorem III*): There exists no smooth deformation retraction from D^{n+1} to S^n, for any n, leaving S^n pointwise fixed.

proof: Suppose $h : D^{n+1} \to S^n$ is such a map. We can view D^{n+1} as the product manifold $S^n \times [0,1]$ and thus view h as a homotopy $h : S^n \times [0,1] \to S^n$ which smoothly deforms any two maps $f, g : S^n \to S^n$ into one another[10] and this contradicts the following degree mod 2 property: the constant map $c : S^n \to S^n$ has degree mod 2 of zero and the identity map $i : S^n \to S^n$ has degree mod 2 of 1. Thus there must not exist a smooth homotopy between c and i but we just said h is one such homotopy. ■

[10] This is not a contradiction to the fundamental group isomorphism theorem.

Degree on orientable manifolds

In this section we briefly mention the generalized formulation of the degree of a map (see Milnor 1972 and Guillemin and Pollack 2010 for proofs) which is occasionally more useful. This generalization requires a further restriction on M and N, namely that they are both orientable manifolds in the following sense.

Let M be any smooth k-manifold. At every point of M there exists a local diffeomorphism f carrying an open set $U \subseteq \mathbb{R}^k$ to a neighborhood $g(U)$ in M. We earlier defined the tangent space TM_x in terms of the image $df_x(\mathbb{R}^k)$. As a linear transformation we can assign an "orientation" to df_x in terms of the sign of its determinant (which is invariant under change of bases). A manifold is then called *orientable* if it has a choice of local diffeomorphisms such that the sign of $\det df_x$ is constant over M.[11]

The *degree* of a map $f : M \to N$ defined over smooth oriented manifolds of equal dimensions where M is compact and without boundary is defined to be

$$\deg(f; y) = \sum_{x \in f^{-1}(y)} \operatorname{sgn} \det df_x$$

for any regular value $y \in N$ of f. The central result is then exactly the same as the result for degree mod 2:

Theorem: The degree of a mapping defined as above is independent of the choice of regular value y and is invariant under smooth homotopies.

Example: The simplest case to study the degree is when $f : M \to S^n$ in which case the degree is more appropriately referred to as the *winding number*. In the simplest case when $n = 1$ we can informally convince ourselves that the degree is precisely equal to the number of times f winds around the circle (note that since the circle is path-connected the fundamental group is independent of choice of base point) and this correspondence is made more clear when we refer to our characterization of $\pi_1(S^1) = \mathbb{Z}$.

The above theorem says that any two homotopic maps over manifolds satisfying the conditions of the theorem must have equal degrees. This is yet another tool to establish impossibility of certain continuous maps (here a continuous deformation of a map to another) as demonstrated in our third variation of the non-retraction theorem.

A natural question arises: does the above have a converse?

[11] It follows from the sgn det df_x construction that any such manifold M has precisely two possible orientations.

That is, are two smooth maps with identical zeros necessarily homotopic?

There is a converse in the following sense which we mention without proof (see Guillemin and Pollack 2010 for a proof of a stronger theorem):

Theorem (*Hopf degree theorem*): Two maps of a compact, connected, oriented k-manifold M into S^k are homotopic if and only if they have the same degree.

Index theorems

The notion of indices naturally arises in many types of questions involving vector fields. In all cases, the index has (as a topological property) global invariant properties and as such imposes topological restrictions. In the second part we will see two modified index theorems suited to specific problem domains. In this section we look at the classic Poincaré-Hopf theorem. First, we define the index in terms of the degree of a mapping and then outline the proof of our main theorem following (Milnor 1972).

Let $v : M \to \mathbb{R}^m$ be a smooth vector field on a manifold M. Consider an arbitrary isolated zero z of v. Some neighborhood U of z is mapped by

$$g(x) = v(x)/|v(x)|$$

to S^m. The degree of $g : U \to S^m$ is the *index of* v *at the isolated zero* z. Symbolically, $\iota_z(v)$ is defined to be $\deg(g)$.

Theorem (*Poincaré-Hopf*[12] *index theorem*): Let M be a compact manifold and v be a smooth vector field over M with isolated zeros.[13] The sum of indices of v at its zeros equals the Euler characteristic $\chi(M)$:[14]

$$\sum_{z \text{ v(zero)=0}} \iota_z(v) = \chi(M)$$

Most significantly, the right hand side is independent of the choice of vector field v.

proof: A complete proof requires quite a lot of work. Here we sketch the outline of the proof in (Milnor 1972) as it provides insight into alternative more useful formulations of the theorem.[15] The following lemmas give an intuitive overview of what the result entails. In all of them we assume M, and v are as in the theorem:

Lemma: Any orientation preserving diffeomorphism $f : \mathbb{R}^m \to \mathbb{R}^m$ is smoothly homotopic to the identity.

Lemma: A map $f : M \to N$ which carries the vector field v over M diffeomorphically into the vector field v' over N does not change the index at any of the isolated zeros of v. That is, if a zero z is carried to z' by f we have: $\iota_z(v) = \iota_{z'}(v')$.

Lemma: At any nondegenerate zero z we have $\iota_z(v) = \operatorname{sgn}\det dv_x(z)$.

Remark: Specifically, this allows us to write the following in terms of π and μ of each nondegenerate zero (which we already discussed in the classification of linear systems). Since we know that at a nondegenerate zero:

$$\operatorname{sgn}\det dv_x(z) = (-1)^{\mu_z}$$

where μ_z is the number of eigenvalues of the Jacobian map of v at z with negative real parts. Therefore, we can write:

$$\chi(M) = \sum_{z \text{ isolated zero}} (-1)^{\mu_z}$$

Remark: The theorem makes no distinction of the degeneracy of zeros. The case of degenerate zeros must be dealt with separately.

Lemma: (*Hopf's lemma*) The index sum of indices of any v with isolated zeros is equal to $\deg(g)$ where $g : \partial M \to S^{m-1}$ is the *Gauss mapping* which assigns to each point x on the boundary of M the outward unit normal vector at x. Most significantly, the sum of indices is independent of the choice of v.

Given the above lemma, it suffices to find any vector field over M that makes it easy to demonstrate that its sum of indices is $\chi(M)$. One such characterization can be done using Morse's gradient fields (Milnor 1972). ■

Remark: The Poincaré-Hopf theorem has a differential geometric analog, the *Gauss-Bonnet theorem*, which also involves $\chi(M)$, that characterizes the index in terms of an integer multiplier of the integral of Gaussian curvature around the boundary of a Riemannian manifold. In this sense, the Poincaré-Hopf theorem is a higher dimension generaliza-

[12] The Hopf in this and the next theorem refers to Heinz Hopf, not the same person as Eberhard Hopf, of the Hopf bifurcation.

[13] If M has a boundary we additionally demand that v points *outwards* everywhere on the boundary. In what follows we do not need this requirement as we will be mostly concerned with manifolds without boundary.

[14] We also know that for any closed orientable manifold the Euler characteristic χ and the genus g satisfy the relation $\chi = 2 - 2g$. Additionally, for compact 2-manifolds, any triangulation of M satisfies $\chi = V - E + F$ where V is the number of vertices, E is the number of edges, and F is the number of faces (i.e polygons). A similar generalized result applies to higher dimension manifolds.

[15] In (Guillemin and Pollack 2010) the theorem is proved using the notion of *transversality* and the Lefschetz theorem.

tion. However, in many cases, it is easier to apply the familiar integration techniques to *derive* new index theorems as we will see in the next part.

Part II: applications

In this part we look at a family of results in biological and chemical dynamics with one common theme: impossibility of continuous maps which we have discussed in detail in various sections of part I.

Phase maps

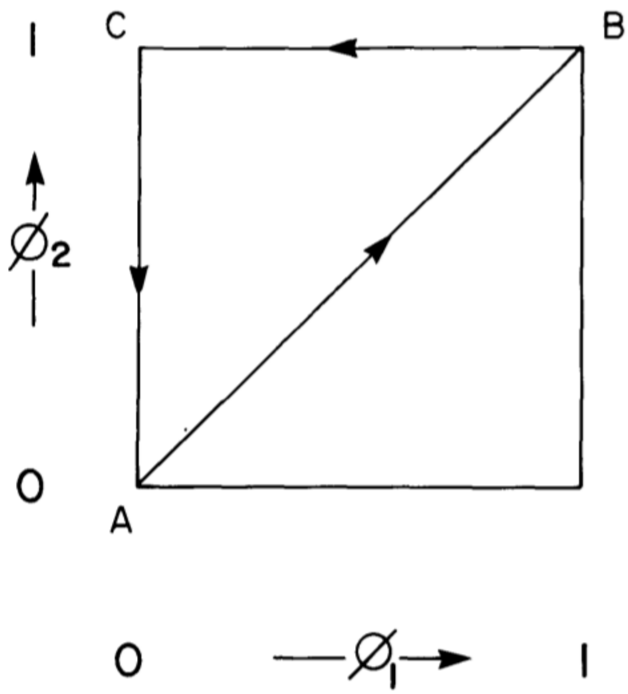
The circle S^1 and its higher dimension counterparts are the natural phase space for typical oscillating phenomena. Consider any population of oscillating systems (we will typically be thinking of the population as a manifold in the space since the medium may be continuous such as a petri dish of chemical reactants). As time goes on various parts of the population go through various phases of periodicity, presumably in some sort of harmony with the rest of population. A phase map is an assignment of phase, which we have not yet defined, to each of the members of the population. By phase, we mean the "renormalized time" (author's term) of each cell with respect to its periodicity. Here are some examples (Winfree 2001):

1. *Tides*: without requiring any detail about the mechanism, we simply observe that the phenomenon is periodic in the sense that every point on earth is subject to a repeating succession of qualitatively recognizable stages. One can pick any of the components of the phenomena (again, without regard to decouplable complexities, like harmonics of tidal waves and sensitivities to local shoreline geometry) and assign to each point on the surface of the earth a *phase*, a point on $\phi \in S^1$ corresponding to the position of an imaginary oscillator with the same period on a S^1. It is evident that this is a better choice than $\phi \in [0, 1]$ since it avoids giving the appearance of non-existent discontinuities of phase as the periodic observable winds around S^1 and returns to its "starting value". By defining ϕ everywhere on the surface of the earth we have in effect built a map $f : S^2 \to S^1$ which is what we refer to as a *phase map*.
2. *Glycolytic oscillations* are observable from various metabolite concentrations. In a population of yeast (e.g. a petri dish corresponding to D^2 or a 3 dimensional volume with a boundary) we can define in the same fashion a map which carries each point of the medium, without us requiring knowledge of exact biochemical relationships, to a point on the circle corresponding to position of each oscillator (a yeast cell) along its own period given an agreed upon "start phase" everywhere. For example, one can say the "start phase", i.e. $\phi = 0$, of each cell is the peak concentration point in time of phosphofructokinase, a key glycolytic metabolite. Such a map is again a phase map, say $f : D^2 \to S^1$.
3. *Circadian rhythms* of fruit flies: suppose we take for granted that somehow the circadian rhythm is synchronized to patterns of exposure to light. Then one can decide, and the choice is arbitrary, that the start phase is the last exposure to extended darkness. Now suppose we gather a population of pupae and spread them on a table (that is $[0, 1] \times [0, 1]$, homeomorphic to D^2), in a room under constant light. We wish to perform an experiment (we do not care here what experiment, (Winfree 2001) elaborates on the experiment the author performed and the results he obtained) where we manipulate the light exposure of different regions of the table by an artificial, moving shadow. At any time, there is a *phase map* assigning to each point on D^2 a point on S^1 corresponding to the *phase* of each individual in their circadian cycle (which varies by different patterns of exposure across the table).
4. *Circulating waves* in an excitable medium (a nerve fiber, or a muscle ring in the heart) subject each point of the medium to a succession of landmarks (Winfree 2001): "rest", "excited", "refractory" and potentially intermediate values (note that, relatively accurate measurements of cell membrane potentials of neurons are experimentally done routinely but we do not need any of this information). This, similar to above examples, imposes a phase map $f : S^1 \to S^1$. Similarly, for a 2-dimensional medium of BZ reagent the phase map carries D^2 to S^1 and a for a 3-dimensional volume the phase map carries some manifold with boundary M to S^1.
5. A simple *phase resetting* experiment: consider the glycolytic yeast medium from a previous example. Suppose we abruptly combine two populations of yeast, each synchronous on their own with phases ϕ_1 and ϕ_2. It has been experimentally verified that the combined population approaches a new phase ϕ (after transients) which is an intermediate "compromise" phase. Suppose we were to study the behavior of such a *phase resetting* experiment. What we wish to study is a *resetting map* taking $S^1 \times S^1$ (i.e. the torus) to S^1.

Impossibility of continuous maps

The above examples are all subject to arguments of the style of the non-retraction theorem we proved in the previous part in various forms (Strogatz 1985; Winfree 2001; Winfree and Strogatz 1983). For example, in Strogatz 1985, the following seemingly conservative expectations are posed for the yeast glycolytic phase resetting experiment: *i.* "interchanging the names of groups 1 and 2 will not affect the outcome *ii.* if the separate groups agree in phase initially, they continue at that phase, unaffected by mix true with their own kind. *iii.* Slight changes in the initial phases alter the outcome only slightly."

This, only to demonstrate, by a winding number argument, that such a continuous function may not exist.[16] Notice that our assumption is demanding two additional conditions beside continuity: symmetry and that f be such that (condition ii) $f(\phi, \phi) = \phi$ for any $\phi \in S^1$. This implies that the image, under the phase map, of the loop over the torus characterized by $\phi_1 = \phi_2$ winds around S^1 once and thus has degree 1. But the equivalent loop consisting of two parts $\phi_1 = 0$ and $\phi_2 = 0$ has an even degree which is impossible.



Here we demonstrate the point using fundamental groups:

Proposition: There exists no continuous function $f : S^1 \times S^1 \to S^1$ such that it is: 1. symmetric in its arguments, 1. for any $\phi \in S^1$ we have $f(\phi, \phi) = \phi$.

proof: Suppose such a function it exists. It must then induce a homomorphism f_* between their corresponding fundamental groups, the correspondence given by the fact that f carries every loop in T^2 into a loop in S^1. We know that $\pi_1(S^1) = \mathbb{Z}$ and $\pi_1(T^2) = \mathbb{Z}^2$. The latter means this: the loop ℓ corresponding to one whole revolution of ϕ_1 and ϕ_2 around S^1 constrained by $\phi_1 = \phi_2$ and let the loops $\ell_{1,2}$ each making one of ϕ_1 wind around S^1 once while keeping the other constant. Then we have ℓ is homotopic to $\ell_1 * \ell_2$. By the symmetry requirement we must have $f_*([\ell_1]) = f_*([\ell_2])$ and therefore $f_*(\ell)$ must be an even integer. But the second requirement forces this loop to be mapped to 1 which is a contradiction. ■

Similarly, consider the BZ reaction in a two dimensional medium (and the identical argument to the phase map of the table full of young pupae). By the non-retraction theorem, the phase map $f : D^2 \to S^1$ cannot possibly be continuous and therefore there must be at least one point of discontinuity of f somewhere in the medium. We will return back to this experiment as the basis of spiral waves in an excitable medium.

A similar argument applies to the timing of tides described above. The phase map carries the surface of the earth, homeomorphic to S^2, to the circle which is not itself forbidden but is inconsistent, degree wise, with the observation that in a full circle around the earth at any time the phase must finish two full cycles (as the high tide and low tide loci come in antipodal pairs) (Winfree 2001). Here is again a more succinct version using the fundamental group:

Proposition: There is no continuous map $f : S^2 \to S^1$ that takes the equator to a loop winding twice around S^1.

proof: The fundamental group of S^2 is trivial and the fundamental group of S^1 is $\mathbb{Z}$. Therefore, the induced homomorphism f_* of such a continuous map must carry all loops on the sphere into the identity element of S^1 which is the homotopy class of the constant loop ℓ corresponding to the winding $0 \in \mathbb{Z}$ around S^1. This is not possible while demanding that the equator loop in S^2 is carried to loop with winding twice about S^1. ■

Singularity filaments in the BZ reaction

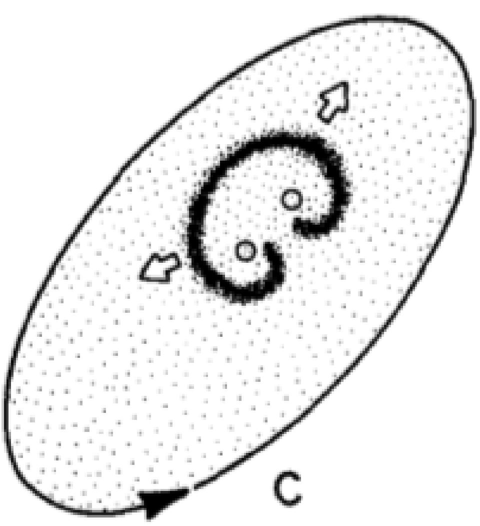
In the discussion above we only established that certain continuous functions may not exist. We did not discuss what

[16]The non-retraction results do not apply here as far as I understand. Strogatz does elude to the non-retraction theorem however but proves his point by a degree argument.

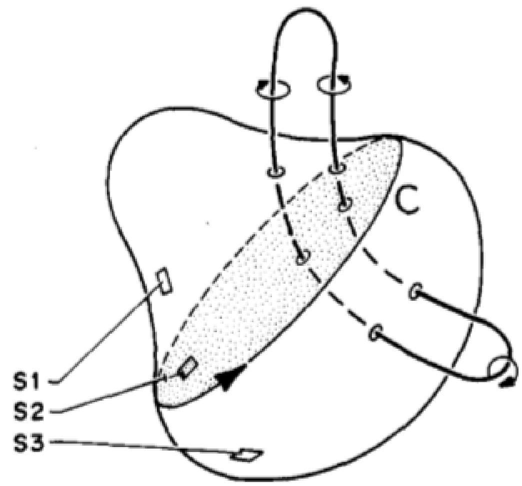
actually happens to the map that must exist, even if it is discontinuous. In this section we follow (Winfree and Strogatz 1983) which is the first article in a series of articles which describe a topological model of the three dimensional spiraling wave in the BZ reaction based only on satisfying the known topological constraint of the corresponding manifolds. Here we outline the main arguments:

As we already demonstrated there must be a singularity of some sort, referred to as a *phase singularity* in the phase map of the BZ oscillator in a two dimensional medium. Now consider a loop in the medium (which we assume is the open ball instead of D^2 since special care is needed for treating phase singularities that arise or annihilate on the boundary). Let the loop be so far from any disturbance that along the loop the entire medium is quiescent (and thus has constant phase). This loop is mapped by the phase map to the constant loop in S^1.

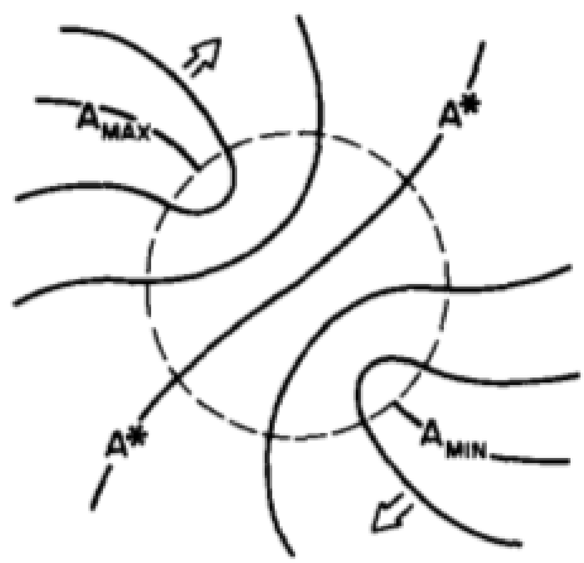
Suppose we isolate a single singularity in a closed loop. By observation of pattern of the spiral waves, we know that such a loop must be carried by the phase map to a loop that winds about S^1 once clockwise or counterclockwise. As the interior of our first large loop is disturbed singularities arise but always such that the loop is still mapped to the constant loop in S^1. If we excise neighborhoods of each arising singularity, the previous argument tells us that singularities must arise in pairs of clockwise and counterclockwise rotating cores.



Now consider a three dimensional volume (that is a 3-manifold with boundary). Again ignoring the boundary for simplicity the same argument as above applies to infinitesimally close cross sections of the volume. The spatial position of singularities in adjacent cross sections can be brought arbitrarily close to each other. Therefore, we would guess by the preceding arguments that phase singularities lie on a looped 1-manifold within the excitable volume (the *singularity filaments*).



Here is another way of arriving at the same idea of singularity filaments: In the 2 dimensional case, variation of phase around any loop enclosing a single rotor must occur monotonically such that in a single wind of the loop the phase comes back to its original value on S^1. Therefore, the average concentration contour passes through the center of the rotor:



It is therefore reasonable to *define* the locus of the center of the rotor to be the intersection point of contours of constant concentration with the average corresponding concentration for each species.

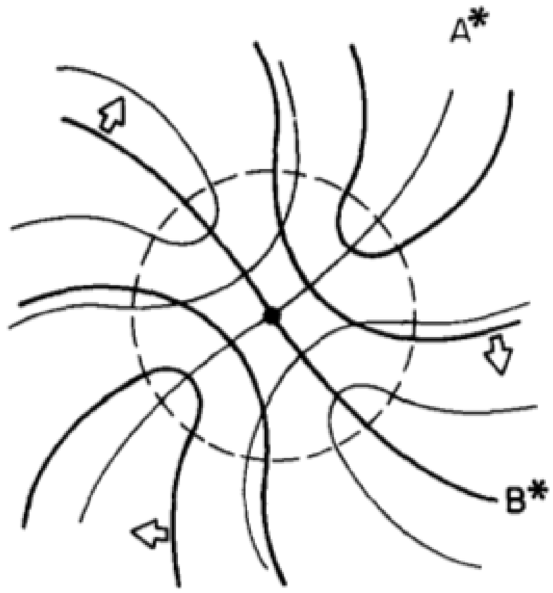


Figure 7: Intersection of average concentration contours as the center of the rotor (from (Winfree and Strogatz 1983))

Now returning back to the 3 dimensional medium, extending the above argument yields the geometric locus of phase singularities as the intersection of the two manifolds corresponding to the surfaces of constant concentration one for the average concentration of each species. This is precisely the singularity filaments the earlier arguments postulated:

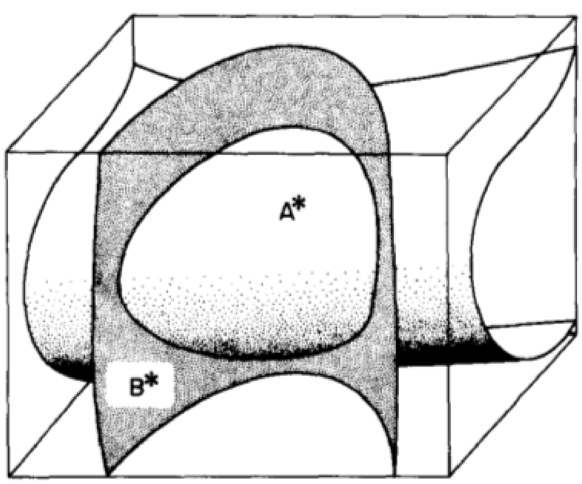


Figure 8: Singularity filaments as the intersection of constant concentration surfaces with average concentration (from (Winfree and Strogatz 1983))

Excursion: A natural question is: what is the geometric shape of the scrolling waves? Our first guess, based on looking at the petri dish for not too long, may be the Archimedean spiral. Another model is the circle involute. The two models have significant differences in their description of the core of the phase singularity (called the *rotor*): the Archimedean spiral is such that all phase contours must converge into a point (this already poses a topological difficulty, see Winfree 2001) but the involute circle model is such that spirals do not get closer than a radius to the core. Additionally, the direction of propagation at the core for the Archimedean spiral is far from radial whereas the involute circle has the property that the direction of propagation obey Huygen's principle: the direction of propagation is always perpendicular to the wavefront.

Alternative index theorems

In this final section we look at an index theorem that is tailored to specific biological/chemical phenomena. This result (Glass 1975) is an index theorem[17] for the class of dynamical systems for which there exists a inner and an outer trapping region. That is a small ball m in the phase space and a large ball M such that the flow of the dynamical system is always inwards on M and outwards on m. Such a requirement is typically met in ecological and chemical networks (exceptions do exist, for example invasion/extinction models do not have the corresponding small ball m and the following theorem does not hold).

Proposition: Suppose the trapping balls m and M exist for an n-dimensional dynamical system. Then

$$\sum_{i=1}^{n}(-1)^{\pi_i} = 1$$

where π_i as before is the number of eigenvalues of the Jacobian at fixed point labeled with i that have positive real parts.

proof: Note that the annular trapping region can be embedded (Guillemin and Pollack 2010) in the n-sphere with the repellent boundary identified as an unstable source at either of the poles. The Euler characteristic of the embedding space (which is S^n) is $\chi = 1 + (-1)^n$. The Poincaré-Hopf theorem written over this region, as we have already paraphrased it earlier, says:

$$\sum_{i=1}^{n+1}(-1)^{\mu_i} = \chi$$

[17] The approach of proving modified index theorems for specific phenomena has also been pursued in (Davidsen, Glass, and Kapral 2004) gives an index theorem for phase singularities we discussed above over a punctured sphere - which is important, say in the heart- and in (Glass 1977) which applies an index theorem to justify peculiar limb regeneration experimental data.

Where one of the fixed points in the sum is an unstable node placed at the south pole corresponding to the inward flow of the dynamics which has $\mu_i = 0$. The result now follows by a rearrangement and noting that $\pi_i + \mu_i = n$ for all eigenvalues. ■

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