Techniques for Inhomogeneous Linear PDEs

Green's Functions and Generalizations of Duhamel's Principle

Green's functions and Duhamel's principle are both standard techniques for solving inhomogenous linear PDEs. Typically the former is presented as a technique for solving boundary value problems for spatial PDEs (e.g. the Poisson equation) and the latter as one for initial value evolution PDEs (e.g. the wave or diffusion equations). However, Green's functions are also applicable to evolution equations where initial values can be viewed as boundary values in space-time. In addition, fundamental solutions to evolution equations enjoy a Duhamel-like property whereby one can "move" the Dirac delta inhomogeneity to initial conditions. In this chapter, we begin by presenting general formulations of Duhamel's principle for linear evolution equations of arbitrary order as well as fundamental solutions and Green's functions for arbitrary linear equations, with special attention given to linear evolution equations. For these equations we also look at different ways to obtain Green's functions and discuss how causality is reflected in the structure of Green's functions. We then establish a Duhamel-like principle for finding fundamental solutions for linear evolution operators of arbitrary order, and generalize this to inhomogeneous evolution equations with space-time distributaional source terms that are "time-factorizable". As another generalization of Duhamel's principle, we present a spatial analog for inhomogenous boundary value problems in bounded star domains. Finally, we briefly discuss convergence methods for obtaining solutions to inhomogeneous linear equations from sequences of approximate solutions.

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§ 1 Background

In this section we present our starting point in the familiar form that one encounters in an introductory study of PDEs. We are concerned with inhomogeneous linear PDEs of the following varieties:

$$\begin{cases} Lu = f & \text{on } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^n$ is a, potentially unbounded, open domain, f is either a function with certain regularity properties or merely a distribution, and L is a linear partial differential operator:

$$L = \sum_k a_k(x) D^{\alpha_k}$$

where D^α refers to multi-index notation for partial differentiation:

$$\alpha = (\alpha^1, \ldots, \alpha^n), \qquad |\alpha| = \sum_i^n \alpha^i, \qquad D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha^1} \ldots \partial x_n^{\alpha^n}}$$

Also of particular interest to us are initial value problems for inhomogeneous linear evolution equations of the first temporal order (parabolic) and second temporal order (hyperbolic) where the initial conditions are zero. We note that solutions to an IVP with non-vanishing initial conditions can be obtained using the principle of superposition.

Definition 1.1. *A* ***first order inhomogeneous linear evolution*** *IVP is*

$$\begin{cases} (\partial_t - L)u = f & \text{for } t > 0 \\ u(x,0) = 0 \end{cases}$$

for some linear partial differential operator L over the space variable $x \in \Omega \subset \mathbb{R}^n$.

Definition 1.2. *A* ***second order inhomogeneous linear evolution*** *IVP is*

$$\begin{cases} (\partial_{tt} - L)u = f & \text{for } t > 0 \\ u(x,0) = 0 \\ \partial_t u(x,0) = 0 \end{cases}$$

for some linear partial differential operator L over the space variable $x \in \Omega \subset \mathbb{R}^n$.

In either case Ω is a, potentially bounded, open domain in $\mathbb{R}^n$ ("physical" space), and u and f are functions over space-time $\mathbb{R}^n \times [0, \infty)$. We keep in mind the inhomogeneous diffusion equation as an important representative for 1.1 and the inhomogeneous wave equation as one for 1.2.

§ 1.1 Duhamel's principle for evolution equations

In standard introductory treatment, Duhamel's principle is presented as a technique to solve the inhomogeneous diffusion and wave equations. The proof of the principle simply follows from linearity of the PDE and an application of elementary results in calculus. Here we present the general form of the standard Duhamel's principle for arbitary linear evolution equations.

▶ **Theorem 1.1** ***First Order Duhamel's Principle*** *Concerning problem 1.1, define auxiliary equations*

$$\begin{cases} (\partial_t - L)u^s = 0 & \text{for } \; t > s \\ u^s(x, s) = f(x, s) \end{cases} \tag{2}$$

where s is a time-like parameter varying over $(0, \infty)$. If the above family of equations have solutions $u^s(x, t)$ then the function

$$u(x, t) := \int_0^t u^s(x, t) ds$$

solves 1.1.

Duhamel's principle extends readily to higher order evolution equations. Here we prove the second order case as a template.

▶ **Theorem 1.2** ***Second Order Duhamel's Principle*** *Consider the set of auxiliary equations for problem 1.2:*

$$\begin{cases} (\partial_{tt} - L)u^s = 0 & \text{for } \; t > s \\ u^s(x, s) = 0 \\ \partial_t u^s(x, s) = f(x, s) \end{cases} \tag{3}$$

where s is a time-like parameter varying over $(0, \infty)$. If the above family of equations have solutions $u^s(x, t)$ then the function

$$u(x, t) := \int_0^t u^s(x, t) ds$$

solves 1.2.

Proof. We apply the components of the evolution operator $\partial_{tt} - L$ to u separately. For the spatial component L we simply have:

$$Lu = L\int_0^t u^s(x,t)ds = \int_0^t Lu^s(x,t)ds$$

since L only applies to x which is fixed in the integrand. For the temporal component we have

$$\partial_t u = \partial_t \int_0^t u^s(x,t)ds = u^t(x,t) + \int_0^t \partial_t u^s(x,t)ds$$

by an application of the fundamental theorem of calculus and the total derivative formula. Noting that $u^t(x,t) = 0$ by the imposed initial conditions we can proceed to the second derivative:

$$\partial_{tt} u = \partial_t \int_0^t \partial_t u^s(x,t)ds = \partial_t u^t(x,t) + \int_0^t \partial_{tt} u^s(x,t)ds = f(x,t) + \int_0^t \partial_{tt} u^s(x,t)ds$$

Combining this with the spatial component and observing that $(\partial_{tt} - L)u^s = 0$ for all $s < t$ completes the proof.

□

The above proof carries over seamlessly to linear evolution equations of arbitrary order

Definition 1.3. ***Inhomogeneous Linear Evolution Equation of order*** k *is an IVP of the form*

$$\begin{cases} (\partial_t^k - L)u = f & \text{for } t > 0 \\ \partial_t^l u(x,0) = 0 & \text{for } l < k \end{cases}$$

▶ **Theorem 1.3** ***General Duhamel's Principle for Linear Evolution Equations*** *Consider the general inhomogenous linear evolution IVP 1.3. Set the auxiliary equations indexed by $s \in (0,\infty)$*

$$\begin{cases} (\partial_t^k - L)u^s = 0 & \text{for } t > s \\ \partial_t^l u^s(x,s) = 0 & \text{for } l < k-1 \\ \partial_t^{k-1} u^s(x,s) = f(x,s) & \end{cases} \tag{4}$$

If the above family of equations have solutions $u^s(x,t)$ then the function

$$u(x,t) := \int_0^t u^s(x,t)ds$$

solves 1.3.

§ 1.2 Diffusion (heat) kernel

Definition 1.4. *The* ***heat kernel*** $\Phi : \mathbb{R}^n \times (0, \infty) \to \mathbb{R}$ *is the function*

$$\Phi(x,t) := \frac{1}{(4\pi t)^{n/2}} \exp\left(-\frac{|x|^2}{4t}\right)$$

The heat kernel has the following standard properties which we will accept without proof.

▶ **Theorem 1.4** ***Properties of the heat kernel***

1. *For all* $x \in \mathbb{R}^n$ *and all* $t > 0$ *the heat kernel satisfies* $(\partial_t - \Delta)\Phi(x,t) = 0$.
2. *As* $t \to 0^+$ *we have* $\Phi(x,t) \to \delta(x)$ *in the sense of distributions.*
3. *The solution to the homogeneous diffusion IVP with initial conditions* $u(x,0) = g(x)$ *is given by the convolution integral*

$$u(x,t) = \int_{\mathbb{R}^n} \Phi(x-y,t)g(y)dy$$

One also encounters the term *fundamental solution for the diffusion equation* for the heat kernel. However, as we shall see, fundamental solutions are a broader concept and this equivalence, although accurate, is a special property of certain linear evolution equations including the diffusion equation.

§ 1.3 Fundamental solutions of Δ

Definition 1.5. *The fundamental solution of* Δ *is defined to be*

$$\Phi(x) := \begin{cases} \frac{1}{2\pi}\log|x| & \text{if } n = 2 \\ -\frac{w_n}{4\pi|x|^{n-2}} & \text{if } n > 2 \end{cases}$$

for $x \neq 0$, *where* n *is the number of space dimensions,* w_n *is a constant equal to the inverse of the hypersurface area of the unit ball in* $\mathbb{R}^n$.

The fundamental solution of Δ has the following properties.

▶ **Theorem 1.5** ***Properties of the fundamental solution of*** Δ

1. $\Phi(x)$ *is harmonic for all* $x \neq 0$, *i.e.* $\Delta\Phi(x) = 0$.
2. $\Delta\Phi(x) = \delta(x)$ *in the sense of distributions.*

3. *(Convolution property) A solution to the Poisson equation $\Delta u = f$ is given by the convolution*

$$u(x) = \int_{\mathbb{R}^n} \Phi(x-y)f(y)dy$$

4. *(Representation formula) If u is harmonic everywhere in Ω we have*

$$u(x) = \int_{\partial\Omega} \Big[u(y)\partial_n \Phi(x-y) - \Phi(x-y)\partial_n u(y)\Big] dS_y$$

for all $x \in \Omega$ where ∂_n is the derivative in the outward normal direction to $\partial\Omega$.

As we shall see in the next section, properties 1 and 2 in theorem 1.5 form the basis for the *definition* of fundamental solutions of arbitrary linear differential operators. Property 3 will then be a simple, yet crucial, consequence of property 2. The representation formula (property 4), however, is unique to Δ as it relies on Green's second identity which does not hold for arbitrary differential operators. A consequence of the representation formula is our ability to use Green's functions to construct solutions for the Dirichlet problem for Laplace's equation. We will see that Green's functions are generally only applicable for solving inhomogeneous equations (for Δ that would be the Poisson equation $\Delta u = f$). This feat of Green's functions to help solve Dirichlet problems for homogeneous PDEs is this a special feature of Δ.

§ 2 Fundamental solutions

In this section we define fundamental solutions for arbitrary linear differential operators and see how they can be used to solve inhomogeneous PDEs with or without boundary conditions. Crucial to all these definitions is the notion of the adjoint of a differential operator used to define distributional derivatives by moving derivatives over to smooth test functions under integrals.
Definition 2.1. ***Adjoint of an Operator*** *Given a linear partial differential operator*

$$L = \sum_k a_k(x) D^{\alpha_k}$$

its adjoint is defined to be

$$L^* = \sum_k (-1)^{|\alpha_k|} a_k(x) D^{\alpha_k}$$

Clearly, for any L we have $(L^*)^* = L$.

Example 2.1 Some adjoint operators

- The adjoint operator for the transport operator $\partial_t - c\partial_x$ is the another transport operator with the same velocity $-(\partial_t - c\partial_x)$.
- The Laplacian Δ and the wave operator $\Box = \partial_{tt} - \Delta$ are self-adjoint.

- The adjoint of the diffusion operator $\partial_t - \Delta$ is the backwards diffusion operator $-\partial_t - \Delta$.
- For any admissible function $f(x)$ we have $Lf(-x) = L^* f(x)$.

The adjoint operator is defined in such a manner to allow partial distributional derivatives to coincide with normal derivatives when a distribution is a normal function. For any linear partial differential operator L, and an appropriately differentiable function u [1], an integration by parts argument shows that

$$\forall \phi \in C_c^\infty(\Omega) \quad \langle Lu, \phi \rangle := \int_\Omega Lu\phi dx = \int_\Omega uL^*\phi dx =: \langle u, L^*\phi \rangle$$

By extension, the distributional application of L on u is *defined* to be the distribution

$$\phi \mapsto \langle Lu, \phi \rangle := \langle u, L^*\phi \rangle := \int_\Omega uL^*\phi dx$$

§ 2.1 Fundamental solutions for linear operators

Definition 2.2. ***Fundamental Solution*** *Given a linear partial differential operator L over $\Omega \in \mathbb{R}^n$, a fundamental solution for L is a parametric family of functions $\Phi^y(x)$, alternatively denoted elsewhere as $\Phi(x, y)$, indexed by a space-like parameter y ranging over the same domain Ω as x, satisfying $L\Phi^y(x) = \delta^y(x)$ in the sense of distributions, that is*

$$\forall \phi \in C_c^\infty(\Omega) : \quad \langle \Phi^y, L^*\phi \rangle = \phi(y)$$

We say that $\Phi^y(x)$ and $\delta^y(x)$ are "centered" at y and for convenience we write $\Phi(x) := \Phi^0(x)$ (justfitication to be provided in theorem 2.1).

Remark. Given a differential operator L fundamental solutions are not unique. That is for a fixed center y there are many functions $v(x)$ satisfying $Lv(x) = \delta^y(x)$. A good (and accurate) analogy is to think about elementary linear algebra: if L was a matrix and u and f were vectors, solutions to $Lu = f$ need not be unique. However, if u and v are such that $Lu = Lv = f$ then by linearity of L difference $u - v$ is in the *kernel* of L, namely $L(u - v) = 0$. In our case, L is a linear transformation on an infinite dimensional vector space, elements of which are differentiable functions. Therefore, we can not really speak of "the" fundamental solution but rather "a" fundamental solution which itself is a parametrized family of functions. We can safely dispense with the pedantry once we show the translation property of fundamental solutions, that "$\Phi^y(x) = \Phi^0(x - y)$" in some sense (see theorem 2.1).

The key observation underlying the utility of fundamental solutions is the fact that for any

[1] to be exact, u needs to be differentiable to $\max_k |\alpha_k|$ order where α_k are the multi-indices involved in L.

function $f : \Omega \to \mathbb{R}$ we have, by definition of the Dirac delta $\langle \delta^y, f \rangle = f(y)$. Intuititvely, by an integral version of the superposition principle, we can expect that the function

$$u(x) = \int_\Omega \Phi^y(x) f(y) dy$$

satisfies $Lu = f$ in a manner similar to property 3 in theorem theorem 1.5.

We will turn this intuitition in to a proper proof in theorem 2.2. But first, we shall establish an important property of fundamental solutions, that "$\Phi^y(x) = \Phi^0(x-y)$" in some sense.

Consider any L and fix any y. The distributional statement $L\Phi^0(x-y) = \delta(x-y) = \delta^y(x)$ holds by definition. This means

$$\forall y \in \Omega, \forall \phi \in C_c^\infty(\Omega): \qquad \int_\Omega \Phi^0(x-y) L^* \phi dx = \phi(0)$$

By a change of variables $\tilde{x} = x - y$ and letting $\tilde{\phi}(x) = \phi(x+y)$ we get

$$\int_\Omega \Phi^0(\tilde{x}) L^* \tilde{\phi}(\tilde{x}) d\tilde{x} = \tilde{\phi}(y)$$

Therefore we have shown that in the sense of distributions

$$L\Phi^0(x-y) = \delta^y = L\Phi^y(x)$$

We have thus proved the following theorem.

▶ **Theorem 2.1** *For any linear differential operator L and any family of fundamental solutions $\Phi^y(x)$ we have*

$$L\Big(\Phi^0(y-x) - \Phi^y(x)\Big) = 0$$

in the sense of distributions.

Based on this, from now on we will abuse notation and drop the superscript 0 and simply write $L\Phi(x-y) = \delta^y(x)$ keeping in mind that L is a differential operator on x and the statement holds in the sense of distributions for any fixed y. This simplifies matters since we can now drop the parametric family of functions Φ^y and get everything we need from one function $\Phi = \Phi^0$, which is commonly referred to, with abuse of terminology, as a (or even worse, the) fundamental solution. However, we do not mind this too much since referring to Φ without the index y is justified by the theorem 2.1, and the use of "the" is justified by the fact that distinct fundamental solutions differ only by a function in the distributional kernel of L.

Remark. In both examples of the previous section, the fundamental solution for Δ in 1.5 and that of the diffusion operator 1.4 (note that we have not shown that either of those are fundamental solutions in the above sense), the fundamental solution has an additional property;

that $\Phi(x-y) = \Phi(y-x)$ and in fact $\Phi^y(x)$ only depends on $|x-y|$. This is *not* in general true. For instance, take $L = \partial_x$. The fundamental solution is the Heaviside function $\Phi^y(x) = \Phi(x-y) = \mathbb{1}_{x>y}$ which clearly does not satisfy $\Phi(x-y) = \Phi(y-x)$. In this sense the fundamental solution of Δ defined in 1.5 coincides with the general definition of fundamental solutions.

We can show that the symmetry property is reserved for fundamental solutions of self-adjoint operators, for instance Δ. One might reasonably object that the diffusion operator is not self-adjoint while the heat kernel 1.4 does enjoy this property. The key here is that x in the heat kernel is only the spatial component of the domain $\Omega \times [0, \infty)$. In fact, as we shall see, the proper fundamental solution for the diffusion operator, which we will show to be the zero extension of the heat kernel to $t < 0$, does *not* enjoy this symmetry property in its full space-time domain.

We can now easily prove the most crucial aspect of fundamental solutions, the integral superposition property:

► **Theorem 2.2** *Let L be any linear differential operator with fundamental solution Φ and let f be any admissible function. The function*

$$u(x) = \int_\Omega \Phi^y(x) f(y) dy = \int_\Omega \Phi(x-y) f(y) dy$$

satisfies $Lu = f$ pointwise.

Proof. We wish to compute

$$Lu(x) = L \int_\Omega \Phi(x-y) f(y) dy$$

keeping in mind that L is a differential operator acting only on x which only appears under Φ in the integrand. However, since $L\Phi$ has a singularity at $x = y$ we cannot simply move L inside the integral. Instead we first use the commutative property of convolutions to write

$$u(x) = \int_\Omega \Phi(y) f(x-y) dy$$

We can now safely bring L under the integral sign noting that it applies to the variable x for any fixed value of y. We now note the crucial property that applying L to the composition $x \mapsto x-y \mapsto f(x-y)$ is identical to applying L^* to the composition $y \mapsto x-y \mapsto f(x-y)$ (to see why this is true note that for any h we have $Lh(x) = L^* h(-x)$ and use the chain rule). Symbolically we write

$$L_x f(x-y) = L_y^* f(x-y)$$

where by the right hand side we mean L^* applied to the function $y \mapsto f(x-y)$ evaluated at a specific x, y. We can now move the L^* back on Φ to get

$$Lu(x) = \int_\Omega \Phi(y) L_y^* f(x-y) dy = \langle L\Phi^0, y \mapsto f(x-y) \rangle = f(x)$$

□

We have so far shown that the fundamental solution of Δ defined in 1.5 is the proper fundamental solution of the Δ as defined in 2.2 with the adjustment of notation according to theorem 2.1. But what about the diffusion operator? For a proper fundamental solution for $\partial_t - \Delta$ we seek a function $\Phi^{y,s}(x,t)$ such that $(\partial_t - \Delta)\Phi^{y,s} = \delta^{y,s}(x,t)$. But all we have is the heat kernel 1.4, defined only for $t > 0$ and only satisfying the property that as $t \to 0^+$ and as a function of x it tends to $\delta(x)$ as a distribution on x. To simplify notation it suffices to find a function $\tilde{\Phi}(x,t)$ such that $(\partial_t - \Delta)\tilde{\Phi} = \delta(x,t)$. We will now use the properties of the heat kernel (theorem 1.4) to show that its zero extension to all t is the fundamental solution we seek.

► **Theorem 2.3** *The function*

$$\tilde{\Phi}(x,t) := \begin{cases} 0 & \text{if } t < 0 \\ \dfrac{1}{(4\pi t)^{n/2}} \exp\left\{-\dfrac{|x|^2}{4t}\right\} & \text{if } t > 0 \end{cases}$$

is the fundamental solution for the diffusion operator $\partial_t - \Delta$ in the sense of 2.2, namely

$$(\partial_t - \Delta)\tilde{\Phi} = \delta(x,t)$$

Proof. For the purpose of this proof we will let Φ be the heat kernel as defined in 1.4, i.e. the $t > 0$ side of $\tilde{\Phi}$. Furthermore, the proof follows identically if we replace Δ with any other linear differential operator on the space variable. So we will write $L = \Delta$ and provide a more general proof applicable to any first order linear evolution equation.

Pick any test function $\phi \in C_c^\infty(\Omega \times \mathbb{R})$, we wish to show

$$\langle(\partial_t - L)\tilde{\Phi}, \phi\rangle = \phi(0)$$

To do this we apply the two components of $\partial_t - L$ to $\tilde{\Phi}$ separately. First we have

$$\begin{aligned} \langle L\tilde{\Phi}, \phi\rangle = \langle\tilde{\Phi}, L^*\phi\rangle &= \int_0^\infty \int_\Omega \Phi L^*\phi dxdt \\ &= \int_0^\infty \int_\Omega L\Phi\phi dxdt \end{aligned}$$

where in the last step we have used the fact that ϕ has compact support and that $L\Phi$ is well defined and has no singularities in Ω.

For the time derivative we have to manage the fact that $\partial_t\Phi$ does indeed have a singularity at

$t \to 0^+$ and we need to tread with care in swapping the derivative back on Φ.

$$\begin{aligned}
\langle \partial_t \tilde{\Phi}, \phi \rangle = -\langle \tilde{\Phi}, \partial_t \phi \rangle &= -\int_\Omega \int_0^\infty \Phi \partial_t \phi dt dx \\
&= -\lim_{\epsilon \to 0^+} \int_\Omega \int_\epsilon^\infty \Phi \partial_t \phi dt dx \\
&= -\lim_{\epsilon \to 0^+} \int_\Omega \left[\Phi \phi |_{t=\epsilon}^\infty - \int_\epsilon^\infty \partial_t \Phi \phi dt \right] dx \\
&= \lim_{\epsilon \to 0^+} \int_\Omega \Phi(x,\epsilon)\phi(x,\epsilon) dx + \lim_{\epsilon \to 0^+} \int_\Omega \int_\epsilon^\infty \partial_t \Phi \phi dt dx \\
&= \phi(0,0) + \int_\Omega \int_0^\infty \partial_t \Phi \phi dt dx
\end{aligned}$$

where in the last step we have used the property of the heat kernel that as $t \to 0^+$ we have $\Phi(x,t) \to \delta(x)$ in the sense of distributions. Putting the two components back together and using the fact that Φ satisfies the diffusion equation for all $t > 0$ completes the proof.

□

Remark. In order to return completely to the proper notation of definition 2.2 for the space-time operator $\partial_t - \Delta$ we should write the fundamental solution of the diffusion equation as

$$\Phi^{y,s}(x,t) = H(t-s)\Phi(x-y, t-s)$$

where H is the Heaviside function and Φ is the heat kernel 1.4. As such, the integral property of fundamental solutions (theorem 2.2) gives the solution

$$u(x,t) = \int_0^\infty \int_\Omega \Phi^{y,s}(x,t) f(y,s) dy dt = \int_0^\infty \int_\Omega H(t-s)\Phi(x-y,t-s) f(y,s) dy dt \qquad (5)$$

for the unconstrained inhomogeneous diffusion equation $(\partial_t - \Delta)u = f$. Note however that this definition satisfies $u(x,0) = 0$, namely that our choice of fundamental solution happens to satisfy the vanishing initial condition that we will eventually want to satisfy.

Remark. Recall that we originally obtained the heat kernel by demanding properties 1 and 2 of theorem 1.4 and then used the Fourier transform. These requirements amount to a distributional initial value problem in the sense that the initial value is a distribution and its imposition must be defined in terms of a distributional limit. We then used these very same properties, amounting to the simplified IVP, in the proof of theorem 2.3 to show that what we obtained from the distributional homogenous IVP can be used to create a solution for a corresponding distributional inhomogenous problem $(\partial_t - \Delta)u = \delta(x,t)$. This process bears more than superficial resemblence to Duhamel's principle. In theorem 4.1 we will clarify this connection further by proving a distributional version of Duhamel's principle. However, we will see that we need to restrict the distributional inhomogeneity, here $\delta(x,t)$, to a certain class of distributions that we refer to as "time-factorizable". We will also see how the above procedure and the standard

Duhamel's principle are special cases of this generalized principle.

§ 2.2 A Duhamel-like principle for fundamental solutions of evolution operators

In the proof of theorem 2.3 we essentially proved the following general result for first order linear evolution operators.

▶ **Theorem 2.4** ***Fundamental solution for first order linear evolution operators*** *Let L be a linear differential operator and suppose the function $\Phi(x,t) : \Omega \times (0,\infty) \to \mathbb{R}$ satisfies the following two properties:*

- $(\partial_t - L)\Phi = 0$ *for all $x \in \Omega$ and all times $t > 0$.*
- *As $t \to 0^+$ we have $\Phi(x,t) \to \delta(x)$ in the sense of distributions.*

Then the function

$$\Phi^{y,s}(x,t) = H(t-s)\Phi(x-y,t-s)$$

is the fundamental solution of $\partial_t - L$ in the sense of definition 2.2 over the space-time domain.

In fact, a slight modification of the same proof can provide the following result for linear evolution operators of arbitrary order k.

▶ **Theorem 2.5** ***Fundamental solution for general linear evolution operators*** *Let L be a linear differential operator and suppose the function $\Phi(x,t) : \Omega \times (0,\infty) \to \mathbb{R}$ satisfies the following two properties:*

- $(\partial_t^k - L)\Phi = 0$ *for all $x \in \Omega$ and all times $t > 0$.*
- *As $t \to 0^+$ we have $\partial_t^l \Phi(x,t) \to 0$ for all $l < k-1$.*
- *As $t \to 0^+$ we have $\partial_t^{k-1}\Phi(x,t) \to \delta(x)$in the sense of distributions.*

Then the function

$$\Phi^{y,s}(x,t) = H(t-s)\Phi(x-y,t-s)$$

is the fundamental solution of $\partial^k/\partial_{t^k} - L$ in the sense of definition 2.2 over the space-time domain.

Example 2.2 Fundamental solution for the 1-D wave operator We can now use this general recipe to obtain fundamental solutions for the wave operator $\square = \partial_{tt} - c^2\Delta$. First by applying the Duhamel-like recipe of theorem 2.5 we reduce the problem to solving for $u(x,t)$ such that

$$\begin{cases} (\partial_{tt} - L)u = 0 & \text{for} \quad x \in \mathbf{\Omega}, t > 0 \\ u(x,t) = 0 & \\ \partial_t u(x,t) \to \delta(x) & \text{as} \quad t \to 0^+ \end{cases}$$

We can now follow the same recipe as the one used to find the heat kernel, for simplicity we will only consider the case of $n = 1$ space dimensions. By applying the Fourier transform to the space variable the requirements above translate to

$$\begin{cases} \partial_{tt}\hat{u}(\omega,t) + c^2\omega^2\hat{u}(w,t) = 0 & \text{forall } x \in \mathbf{\Omega}, t > 0 \\ \hat{u}(\omega,0) = 0 & \\ \hat{u}(\omega,0) = 1 & \end{cases}$$

which for every fixed ω is an ODE in t that can be readily solved to give

$$\hat{u}(\omega,t) = \frac{1}{c\omega}\sin(c\omega t) = \frac{1}{2c\omega i}\left(e^{icwt} - e^{-icwt}\right)$$

We now know that the inverse Fourier transform of $1/\omega i$ is the Heaviside function $H(x)$ and that the Fourier inverse of multiplication by $e^{a\omega}$ is a shift by a. Therefore, we get:

$$u(x,t) = \frac{1}{2c}\left[H(x+ct) - H(x-ct)\right] = \frac{1}{2c}\mathbb{1}_{|x|<ct}$$

By theorem 2.5 the fundamental solution for the wave operator in one space dimension is

$$\Phi(x,t) = \frac{1}{2c}\mathbb{1}_{|x|<ct}$$

which is, unsurprisingly, what we would have obtained where we to informally apply the d'Alembert formula to the distributional IVP above.

To fully return to the notation of 2.2, working our way back through the simpler notification justified by theorem 2.1, we should write the fundamental solution of the 1-D wave equation as

$$\begin{aligned} \Phi^{y,s}(x,t) &= \frac{1}{2c}\Big[H\Big(x - y + c(t-s)\Big) - H\Big((x-y) - c(t-s)\Big)\Big] \\ &= \frac{1}{2c}\mathbb{1}_{|x-y|<c(t-s)} \end{aligned} \tag{6}$$

§ 3 Green's functions for evolution equations

Returning to the original boundary value problem (1) we need to find an appropriate member of the fundamental solutions, namely Green's functions $G^y(x)$ that satisfy the boundary condition. Once that condition is satisfied, we can build solutions to the inhomogeneous BVP using the same integral property of fundamental solutions (theorem 2.2).

Recall that fundamental solutions are never unique but rather a family of possible candidates satisfying $L\Phi^y(x) = \delta^y(x)$. We do know that much like solutions to finite systems of linear equations different candidates differ only by a function in the distributional kernel of L. To obtain an actual solution to a BVP of the form (1) we need to find one among many of these candidates that satisfies the boundary condition. To this end it suffices to find, for each $y \in \Omega$, a solution to the homogeneous BVP

$$\begin{cases} Lu^y = 0 & \text{on } \Omega \\ u^y = \Phi^y & \text{on } \partial\Omega \end{cases} \tag{7}$$

One can then find the appropriate fundamental solution, referred to as the Green's function $G^y(x)$ for the BVP by setting $G^y = \Phi^y - u^y$. Since Green's functions are fundamental solutions theorem 2.2 still holds and one obtains a solution to (1) via

$$u(x) = \int_\Omega G^y(x) f(y) dy$$

The analog of this for inhomogeneous evolution equations is to solve for

$$\begin{cases} Lu^{y,s} = 0 & \text{for } t > 0 \\ \partial_t^l u^{y,s}(x,0) = \partial_t^l \Phi^{y,s}(x,0) & \text{for } l < k \end{cases} \tag{8}$$

where k is the time order of the equation (e.g. for the diffusion equation $k = 1$ and for the wave equation $k = 2$). We then obtain the Green's function from

$$G^{y,s}(x,t) = \Phi^{y,s}(x,t) - u^{y,s}(x,t)$$

and obtain solutions to the IVP using the same integral property

$$u(x,t) = \int_0^\infty \int_\Omega G^{y,s}(x,t) f(y,s) dy ds \tag{9}$$

In the BVP case we are very much at the mercy of symmetries in the domain Ω in order to be able to solve the homogeneous BVP (7). For evolution equations, on the other hand, we typically have standard recipes to solve (8) and obtain $u^{y,s}$ directly.

§ 3.1 Green's functions for the diffusion equation

We wish to solve for every $y \in \Omega$ and $s > 0$

$$\begin{cases} (\partial_t - \Delta)G^{y,s} = \delta^{y,s}(x,t) \\ G^{y,s}(x,0) = 0 \end{cases} \tag{10}$$

In this case we get lucky since, as per remark 2.1, our choice of fundamental solution already satisfies these properties when it's applied to the source term through theorem 2.2. Specifically, the Green's function for the diffusion equation is precisely the fundamental solution described in (5):

$$G^{y,s}(x,t) = H(t-s)\Phi(x-y,t-s)$$

where H is the Heaviside function and Φ is the heat kernel.

According to the solution formula (9), we get

$$\begin{aligned} u(x,t) &= \int_0^\infty \int_\Omega H(t-s)\Phi(x-y,t-s)f(y,s)dyds \\ &= \int_0^t \int_\Omega \Phi(x-y,t-s)f(y,s)dyds \end{aligned} \tag{11}$$

Remark. We observe that the solution formula derived above is identical to what we would have found if we were to use Duhamel's principle (theorem 1.1): the time integral assembles solutions to auxiliary problems with initial conditions $u^s(x,s) = f(x,s)$ and the space integral is the usual convolution solution to the homogeneous diffusion IVP.

§ 3.2 Green's functions for the 1-D wave equation

In this case we need to do a little more work since we need to cancel out two effects of the fundamental solution at initial time. Specifically we need to solve for

$$\begin{cases} (\partial_{tt} - c^2\Delta)u^{y,s} = 0 \\ u^{y,s}(x,0) = \Phi^{y,s}(x,0) \\ \partial_t u^{y,s}(x,0) = \partial_t \Phi^{y,s}(x,0) \end{cases} \tag{12}$$

Using the definition of the fundamental solution $\Phi^{y,s}(x,t)$ given by (6) we simply need to apply d'Alembert's formula and then use $G^{y,s} = \Phi^{y,s} - u^{y,s}$. The final result is

$$G^{y,s}(x,t) = \frac{1}{2c} H\Big(c(t-s) - |x-y|\Big)$$

We thus get the solution formula

$$\begin{aligned} u(x,t) &= \int_0^{\infty} \int_{\mathbb{R}} \frac{1}{2c} H\Big(c(t-s) - |x-y|\Big) f(y,s)dyds \\ &= \int_0^t \int_{x-c(t-s)}^{x+c(t-s)} \frac{1}{2c} f(y,s)dyds \end{aligned} \tag{13}$$

where we have simplified the time range using the fact that $|x-y| < c(t-s)$ is only possible for $s < t$.

Remark. We observe that the solution formula derived above is identical to what we would have found if we were to use Duhamel's principle (theorem 1.2): the time integral assembles solutions to auxiliary problems with initial conditions $u^s(x,s) = 0$ and $\partial_t u^s(x,s) = f(x,s)$ and the space integral is the usual d'Alembert's formula for the homogeneous wave IVP.

§ 3.3 Green's functions from Duhamel's solution: application to 2-D and 3-D waves

In both examples of linear evolution equations above we obtained identical formulae from applying Duhamel's principle in the appropriate order (see theorem 1.3) and from the Green's integral formula. In hindsight we can see how we could have *guessed* the Green's function for both the diffusion and wave equations by looking at the Duhamel solution formula.

We now use this trick to guess the Green's function for the 3D wave equation by first solving the inhomogeneous problem using Duhamel's principle (theorem 1.2). The Duhamel auxiliary problems for the inhomogeneous wave equation are

$$\begin{cases} (\partial_{tt} - c^2\Delta)u^s(x,t) = 0 & \text{for } t > s \\ u^s(x,s) = 0 & \\ \partial_t u^s(x,s) = f(x,s) & \end{cases}$$

By Kirchhoff's formula the solution is

$$u^s(x,t) = \frac{1}{4\pi c(t-s)^2} \int_{\partial B(x,c(t-s))} (t-s) f(y,s)dS = \frac{1}{4\pi c(t-s)} \int_{\partial B(x,c(t-s))} f(y,s)dS$$

Therefore the solution to the inhomogeneous problem is

$$u(x,t) = \int_0^t \frac{1}{4\pi c(t-s)} \int_{\partial B(x,c(t-s))} f(y,s) dS ds$$

We now need to capture the boundary of the spherical integral in a multiplicative function so the spatial integral can be written over the entire domain.

$$u(x,t) = \int_0^t \frac{1}{4\pi c(t-s)} \int_\Omega \delta^{c(t-s)}\Big(|x-y|\Big) f(y,s) dy ds$$

Noting that $|x-y| = c(t-s)$ is only possible for $s < t$ we can now safely remove the range of the time integral as well to obtain

$$u(x,t) = \int_0^\infty \int_\Omega \frac{1}{4\pi c(t-s)} \delta^{c(t-s)}\Big(|x-y|\Big) f(y,s) dy ds$$

Therefore, we can *guess* the Green's function for the 3D wave equation to be

$$G^{y,s}(x,t) = \frac{1}{4\pi c(t-s)} \delta^{c(t-s)}\Big(|x-y|\Big) \tag{14}$$

which is in fact the correct Green's function as it can be verified that $(\partial_{tt} - \Delta)G^{y,s} = \delta^{y,s}$.

Similarly, for the 2-D wave equation the solution formula based on Duhamel's principle

$$u(x,t) = \int_0^\infty \int_\Omega \frac{1}{2\pi ct\sqrt{c^2(t-s)^2 - |y-x|^2}} H\Big(c(t-s) - |x-y|\Big) f(y,s) dy ds$$

where H is the Heaviside function. We can therefore guess the Green's function for the 2-D wave equation

$$G^{y,s}(x,t) = \frac{1}{2\pi ct\sqrt{c^2(t-s)^2 - |y-x|^2}} H\Big(c(t-s) - |x-y|\Big) \tag{15}$$

§ 3.4 Inferring domains of dependence from Green's functions

Considering the solution formula (11) for the inhomogeneous diffusion equation, we can make the following observation about causality. The solution at position x and time t is affected by the source term f at all positions $y \in \Omega$ but only at times $s < t$. In other words the domain of dependence of $u(x,t)$ on the source f is the cylindrical region $\Omega \times (0,t)$ in space-time.

Similarly, based on the solution formula (13) for the inhomogeneous 1-D wave equation, we observe that the solution at position x and time t is affected by the source term f at those times s and positions y that satisfy $|x-y| < c(t-s)$. In other words the domain of dependence of $u(x,t)$ on the source f is the triangular region (a cone in higher dimensions) centered at x, t and facing backwards in time with slope c in space-time.

In fact this logic can be applied to any Green's function. For instance in the case of evolution equations we can easily see that the solution formula (9)

$$u(x,t) = \int_0^\infty \int_\Omega G^{y,s}(x,t) f(y,s) dy ds$$

implies that $u(x,t)$ only depends on the value of f at those positions and times y, s that fall in the support of $G^{y,s}(x,t)$. Therefore, in general, if we fix x, t and regard $G^{y,s}(x,t)$ as a function of y, s, the domain of dependence of $u(x,t)$ on f is precisely the support of $G^{y,s}$.

$$\begin{aligned} \forall x, t : \text{domain of dependence} &= \text{supp}_{y,s} G^{y,s}(x,t) \\ &= \{(y,s) \in \Omega \times (0,\infty) \text{ s.t. } G^{y,s}(x,t) \neq 0\} \end{aligned}$$

For instance, we can see from the Green's function (14) for the 3-D wave equation that, in agreement with Huygen's principle, the domain of influence for any x, t is the three dimensional surface of a 4-D cone defined by $|x-y| = c(t-s)$ in space-time.

Similarly, from the Green's function (15) for the 2-D wave equation we can conclude that the domain of influence for any x, t is the interior of a 3-D cone defined by $|x-y| < c(t-s)$ in space-time.

§ 4 Two generalizations of Duhamel's principle

§ 4.1 Duhamel's principle for time-factorizable distributions

The first generalization unifies the standard Duhamel's principle and the Distributional Duhamel-like results that we used in the previous section to find fundamental solutions for first order linear evolution equations. For simplicity we will consider the first order variety of this generlized Duhamel's principle and we will see that it contains theorems 1.1 and 2.4 as its special case.

Definition 4.1. ***Time-factorizable spatio-temporal distributions*** *Suppose f is a distribution* *over* $C_c^\infty\Big(\Omega \times (0,\infty)\Big)$ *where* $\Omega \subset \mathbb{R}^n$ *is the space domain*

$$f : \phi(x,t) \mapsto \langle f, \phi \rangle$$

We say that f *is time-factorizable if*

1. *there exists a parameteric family of spatial distributions* f^s *over* $C_c^\infty(\Omega)$ *parametrized by a time-like paramater* s *varying over* $(0,\infty)$,
2. *there exists a purely temporal distribution* T, *and*
3. *for every test function* ϕ *we have* $\langle f, \phi \rangle = \langle T, s \mapsto \langle f^s, \phi(\cdot, s) \rangle \rangle$

In other words, f *operates by first applying all spatial distributions* f^s, *for all* $s > 0$, *to spatial functions* $\phi(\cdot, s)$ *for fixed* s. *Then the single variable function* $s \mapsto \langle f^s, \phi(\cdot, s) \rangle \rangle$ *is fed to* T *to produce the final output of* f.

Example 4.1 The spatiotemporal Dirac delta $\delta(x,t)$ is time-factorizable. Let $T = \delta(t)$ and let all spatial distributions f^s be identical to $\delta(x)$, thus producing the temporal map $s \mapsto \phi(0,s)$. It then follows that for all ϕ

$$\langle \delta(x,t), \phi \rangle = \langle \delta(s), s \mapsto \phi(0,s) \rangle = \phi(0,0)$$

Example 4.2 Distributions induced by ordinary functions are also time-factorizable. Suppose f is a function defined pointwise over space-time. Let $T = 1$ and define f^s to be the distributions

$$f^s : \phi \mapsto \int_\Omega f(x,s)\phi(x,s)dx$$

It follows immediately that

$$\langle f, \phi \rangle = \langle 1, s \mapsto \int_\Omega f(x,s)\phi(x,s)dx \rangle = \int_0^\infty \int_\Omega f(x,s)\phi(x,s)dxds$$

▶ **Theorem 4.1** *Consider the distributional first-order inhomogeneous linear evolution equation 1.1*

$$\begin{cases} (\partial_t - L)u = f \\ \quad u(x,0) = 0 \end{cases}$$

where f *is a time-factorizable distribution as per 4.1. Define the auxilary problems parameterized by a time-like parameter* s *varying over* $(0,\infty)$ *as follows*

$$\begin{cases} (\partial_t - L)u^s = 0 & \text{for } t > s \\ \quad u^s(x,s) \to f^s & \text{as } t \to s^+ \end{cases}$$

where the limit is in the sense of distributions. If all auxiliary problems have a solution u^s *then* *the function*

$$u(x,t) := \langle T, H(t-s)u^s(x,t)\rangle$$

where H is the Heaviside function is differentiable and solves the original inhomogeneous problem in the sense of distributions.

Proof. We simply need to verify that the provided function solves the original PDE. By linearity we have

$$\begin{aligned}
\partial_t u &= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left[\langle T, u^s(x,t+\epsilon)\rangle - \langle T, u^s(x,t)\rangle\right] \\
&= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left[\langle T, u^s(x,t+\epsilon) - u^s(x,t)\rangle\right] \\
&= \lim_{\epsilon\to 0}\langle T, \frac{1}{\epsilon}\left[u^s(x,t+\epsilon) - u^s(x,t)\right]\rangle \\
&= \lim_{\epsilon\to 0}\langle T, \mathbb{1}_{s<t}\frac{u^s(x,t+\epsilon)-u^s(x,t)}{\epsilon}\rangle + \langle T, \frac{1}{\epsilon}\mathbb{1}_{t<s<t+\epsilon}u^s(x,t+\epsilon)\rangle
\end{aligned}$$

Now we note that the first term approaches $\langle T, \partial_t u^s(x,t)\rangle$ in the limit. Now pick any test function ϕ

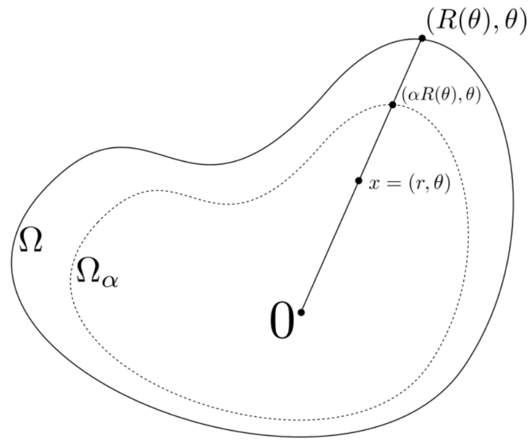
$$\begin{aligned}
\langle \partial_t u, \phi\rangle &= \langle T, \partial_t u^s(x,t)\rangle + \lim_{\epsilon\to 0}\frac{1}{\epsilon}\int_0^\infty \mathbb{1}_{t<s<t+\epsilon}\int_\Omega u^s(x,t+\epsilon)\phi(x,t)dxdt \\
&= \langle T, \partial_t u^s(x,t)\rangle + \lim_{\epsilon\to 0}\frac{1}{\epsilon}\int_{s-\epsilon}^{s}\langle f^s, \phi(x,t)\rangle dt \\
&= \langle T, \partial_t u^s(x,t)\rangle + \lim_{\epsilon\to 0}\fint_{s-\epsilon}^{s}\langle f^s, \phi(x,t)\rangle dt \\
&= \langle T, \partial_t u^s(x,t)\rangle + \langle f^s, \phi(x,s)\rangle
\end{aligned}$$

The spatial operator passes directly through similar integrals and we get:

$$\langle(\partial_t - L)u, \phi\rangle = \langle T, \langle f^s, \phi(x,s)\rangle\rangle = \langle f, \phi\rangle$$

□

We can now see that the standard Duhamel's principle (theorem 1.1) and the distributional generalization we derived in the previous section (theorem 2.4) are special cases of the above result. In each case, the source term is a time factorizable distribution (as seen in examples 4.1 and 4.2).



An illustration of the spatial analog of Duhamel's principle: the point x is where we want to find the solution. The scaled domains Ω_α are those that carry the boundary value for auxiliary problems. Polar coordinates and the star assumption allow us to integrate along the shown line parameterized by the scale parameter α to obtain a solution to the original problem.

§ 4.2 Duhamel's principle for spatial equations in bounded star domains

In this section we present a recipe for using an analog of Duhamel's principle in spatial boundary value problems. The idea is identical to that of temporal Duhamel's but instead of moving the data curve parallel to the time axis we move the data curve "parallel" to the boundary of the original domain. Once auxiliary problems are formulated and solved we can assemble them together using a line integral analogous to Duhamel's temporal principle to get a solution to a linear inhomogeneous BVP.

▶ **Theorem 4.2** ***Spatial analog of Duhamel's principle*** *Suppose $\Omega \subset \mathbb{R}^n$ is a bounded star domain containing the origin and that L is a linear differential operator. Using polar coordinates suppose the unique intersection with $\partial\Omega$ of the line connecting the origin to any point $x = (r, \theta) \in \Omega$ is $(R(\theta), \theta) \in \partial\Omega$. As such, $R(\theta)$ complete characterizes the geometry of the boundary $\partial\Omega$. Consider the boundary value problem*

$$\begin{cases} Lu = f & \text{on } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

Set the auxiliary homogeneous problems parametrized with the one-dimensional parameter $\alpha \in (0, 1)$ to $Lu^\alpha(r, \theta) = 0$ on Ω_α where $\Omega_\alpha = \alpha\Omega$ is the scaled back version of the original domain and additional boundary conditions to be determined from L and f. Then we can set boundary *conditions in such a way that if all the auxiliary problems have solutions then the function*

$$u(r,\theta) = \int_1^{r/R(\theta)} u^\alpha(r,\theta)d\alpha$$

solves the original inhomogeneous BVP.

Instead of a formal proof we provide a recipe that applies to arbitrary linear operators in polar coordinates (r,θ) where r is a scalar and θ is an $(n-1)$-dimensional vector. For simplicity, here we will assume that $n = 2$ and that θ is scalar as well. We fist compute various partial derivatives of u as defined above:

$$\partial_r u = \partial_r \int_1^{r/R(\theta)} u^\alpha d\alpha = \frac{1}{R(\theta)} u^{r/R(\theta)}(x) + \int_1^{r/R(\theta)} \partial_r u^\alpha d\alpha$$

We now note that the first term coincides with the boundary value prescribed to u^α for $\alpha = r/R(\theta)$ at a point which coincides with (r,θ). As we accumulate these terms for various components of L all the second terms add together to form

$$\int_1^{r/R(\theta)} Lu^\alpha d\alpha = 0$$

Therefore, all we need to do is keep track of the boundary terms and arrange boundary conditions accordingly.

Similarly, the angular derivative is

$$\partial_\theta u = \partial_\theta \int_1^{r/R(\theta)} u^\alpha d\alpha = -\frac{\partial_\theta R(\theta) r}{R(\theta)^2} u^{r/R(\theta)}(x) + \int_1^{r/R(\theta)} \partial_\theta u^\alpha d\alpha$$

The first term again involves the boundary condition imposed at u^α for the value of α that makes (r,θ) land on the boundary $\partial\Omega_\alpha$. Therefore, we can replace all occurrences of r in these boundary terms with $\alpha R(\theta)$.

For higher order derivatives we follow the same recipe as Duhamel's temporal principle, namely force all lower order boundary terms to zero and only allow the highest order term to affect boundary conditions. For instance for ∂_{rr} we can force the boundary condition $u^\alpha = 0$ to obtain

$$\partial_{rr} u = \partial_r \int_1^{r/R(\theta)} \partial_r u^\alpha d\alpha = \frac{1}{R(\theta)} \partial_r u^{r/R(\theta)}(x) + \int_1^{r/R(\theta)} \partial_{rr} u^\alpha d\alpha$$

For instance suppose $L = \Delta$ which in polar coordinates is

$$L = \partial_{rr} + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_{\theta\theta}$$

As usual we set the first order boundary condition $u^\alpha = 0$ to get

$$Lu^\alpha = \frac{1}{R(\theta)}\partial_r u^{r/R(\theta)}(x) - \frac{\partial_\theta R(\theta) r}{R(\theta)^2)}\partial_\theta u^{r/R(\theta)}(x) = \left[\frac{1}{R(\theta)}\partial_r - \frac{\partial_\theta R(\theta) r}{R(\theta)^2}\partial_\theta\right] u^{r/R(\theta)}(x)$$

which gives the following formulation of the auxiliary problems:

$$\begin{cases} Lu^\alpha = 0 & \text{on } \Omega_\alpha \\ u^\alpha = 0 & \text{on } \partial\Omega_\alpha \\ \left[\frac{1}{R(\theta)}\partial_r - \alpha\frac{1}{R(\theta)^3}\partial_\theta R(\theta)\partial_\theta\right] u^\alpha = f & \text{on } \partial\Omega_\alpha \end{cases}$$

Remark. If Ω is a spherical domain much of the above calculations simplify since there is no dependence on θ in boundary terms. However, a sphere has ample symmetry for standard methods (e.g. method of inverse for $L = \Delta$) to work. In general however, this method allows on to construct solutions for arbitrary domains for differential operators L that are well understood. For instance if $L = \Delta$ all homogenous BVPs are immediately solvable using the representation formula and this theorem provides a recipe for solving the Poisson equation on arbitrary bounded star domains.

§ 5 Convergence methods

Convergence methods are one of the more useful tools in studying *nonlinear* PDEs. Consider $A(u) = f$ where A is a nonlinear differential operator and f is some function or a distribution. The goal of a convergence method is to come up with a sequence A_n of approximations to A, and a sequence u_n of approximations to u in such a way that in some sense we have $A_n \to A$, and $A_n u_n \to f$. In the context of linear PDEs we can dispense with A_n since a linear operator is already as well behaved as we could wish. We are thus interested in cases where a sequence u_n of functions are such that $Lu_n \to f$ in some sense, and we seek to establish whether u_n converge to some function u, in some sense, such that $Lu = f$?[2]

First, we must assume that L is invertible in the sense that $Lu = 0$ together with appropriate boundary conditions on Ω uniquely defines a solution u in an appropriate function space. This requirement is necessary since if L has a nontrivial kernel then we can always have sequences u_n such that Lu_n converge but the functions u_n are separated by non-vanishing members of the kernel of L.

[2]It is worth noting that the usual bounds on linear operators are not of use here. First, assuming boundedness of L is not helpful since in order to prove convergence of u_n from convergence of Lu_n, what we need is a *lower bound* on the norm of the output of L. Similarly, Poincaré-type inequalities are of no use since (1) the differential operator L might absorb unbounded derivatives of u_n in itself, and (2) even if they do not, these inequalities typically provide *lower bounds* on higher order derivatives which is the opposite of what is needed here.

Second we will also have to assume boundedness of Ω since otherwise (1) useful inequalities do not hold and (2) integration by part is limited to compactly supported integrands.

These two assumptions are the key to proving our two main results. First important consequence is that if we restrict ourselves to a Banach space for candidates u and assume boundedness of Ω then invertibility of L implies invertibility of its adjoint L^*. Second, in a bounded domain we have the nice feature that any smooth solution necessarily has compact support which allows inverting test functions to prove distributional convergence results.

Finally, we note that by linearity of L if u_n happen to converge they must converge to the unique solution of $Lu = f$. So our only goal is to check whether the sequence of approximate solutions do converge in a reasonable sense or not.

▶ **Theorem 5.1** *Suppose L is invertible and Ω is bounded. If the sequence of functions u_n are such that $Lu_n \to f$ in the sense of distributions then u_n converge to the unique solution $L^{-1}f$ in the sense of distributions.*

Proof. Pick any test function ϕ. We wish to show that $\langle u_n, \phi\rangle \to \langle u, \phi\rangle$ where $u = L^{-1}f$. To show this we note that there exists a ψ with compact support such that $L^*\psi = \phi$. Thus we have

$$\langle u_n, \phi\rangle = \langle u_n, L^*\psi\rangle = \langle Lu_n, \psi\rangle \to \langle f, \psi\rangle$$

Therefore, u_n converge in the sense of distributions and we have:

$$\langle u_n, \phi\rangle \to \langle f, (L^*)^{-1}\phi\rangle$$

□

▶ **Theorem 5.2** *Suppose L is invertible, Ω is bounded, and $1 \leq p < \infty$. If the sequence of functions u_n are such that $Lu_n \to f$ in $L^p(\Omega)$ then u_n converge to the unique solution $L^{-1}f$ in $L^p(\Omega)$.*

Proof. This is a consequence of the statement made above: that a bounded linear operator in a Banach space has a bounded inverse. This implies that the inverse of L is continuous in L^p (which is a Banach space for any $1 \leq p < \infty$) which immediately implies the desired result.

□